FIVE-MINUTE OSCILLATION: THEORY

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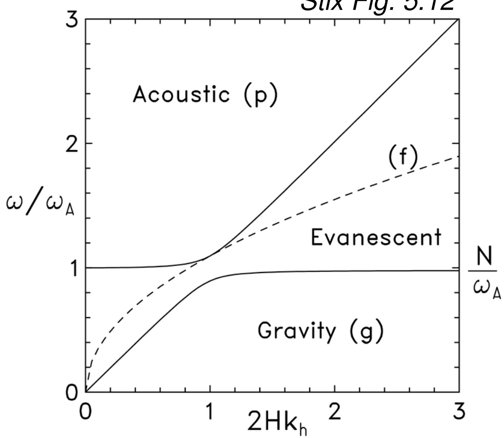
Literature

Jørgen Christensen-Dalsgaard: "Stellar Oscillations", 2003
http://astro.phys.au.dk/~jcd/oscilnotes

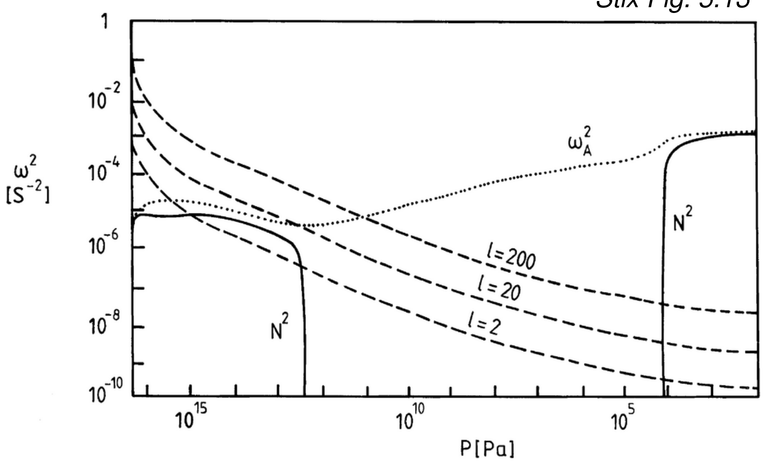
Michael Stix: "The Sun", 2004, second edition, Springer

Rob Rutten: "Fotosferische snelheidsvelden", 1983
http://www.staff.science.uu.nl/~rutte101/Collegedictaat_Fotosferisch.html

Stix Fig. 5.12



Stix Fig. 5.13



BASIC EQUATIONS

Stix sections 2.3.5, 5.2.1

negligible viscosity, perfect conduction, neglect rotation ($\Omega_{\rm rot} \approx 27$ days $\approx 10^4\,\omega_{\rm osc}$)
linear perturbations of hydrostatic equilibrium
subsonic velocities $v \ll c_{\rm s}$ (but chromospheric shocks $M \equiv v/c_{\rm s} \approx 1.1 - 1.5$)

Euler "local" versus Lagrange "material" coordinates

$$[\alpha(t{+}\Delta t)]_{\delta\vec{r}} = \alpha(\vec{r},t){+}\Delta t \left(\left[\frac{\partial\alpha}{\partial t}\right]_{\vec{r}} + \sum_i \frac{\partial\alpha}{\partial x_i}\,\frac{\partial x_i}{\partial t} \right)$$

Langrangian property change: δ

$$\frac{{\rm d}\alpha}{{\rm d}t} = \left[\frac{\alpha(t+\Delta t)-\alpha(t)}{\Delta t}\right]_{\delta\vec{r}} = \frac{\partial\alpha}{\partial t} {+} \vec{v}{\cdot}\nabla\,\alpha$$

first law of thermodynamics

$$\frac{{\rm d}q}{{\rm d}t} = \frac{{\rm d}E}{{\rm d}t} + P\,\frac{{\rm d}V}{{\rm d}t} \qquad V = 1/\rho \qquad \rho\,\frac{{\rm d}q}{{\rm d}t} = \rho\,\frac{{\rm d}E}{{\rm d}t} - \frac{P}{\rho}\,\frac{{\rm d}\rho}{{\rm d}t}$$

ideal gas

$$\delta E = c_v\,\delta T \qquad P = (c_p{-}c_v)\,\rho\,T \qquad P = (\gamma{-}1)\,\rho\,E \qquad \gamma = \frac{c_p}{c_v}$$

ionization: $\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{\rm ad}$

combine

$$\frac{{\rm d}P}{{\rm d}t} = \frac{\gamma P}{\rho}\,\frac{{\rm d}\rho}{{\rm d}t} + (\gamma-1)\,\rho\,\frac{{\rm d}q}{{\rm d}t}$$

adiabatic ($\delta q{=}0$): $\frac{{\rm d}P}{{\rm d}t} = \frac{\gamma P}{\rho}\,\frac{{\rm d}\rho}{{\rm d}t}$

$\delta q \neq 0$: convection (mixing length), radiation (diffusion approx. $\Rightarrow$ Eddington approx. $\Rightarrow$ NLTE)

CONSERVATION EQUATIONS

Stix section 5.2

continuity Euler

$$\int \frac{\partial \rho}{\partial t}\,\mathrm{d}V \equiv \oint -\rho\,\vec{v}\cdot\,\mathrm{d}\vec{s} \stackrel{\text{Gauss}}{=} -\int \nabla\cdot(\rho\,\vec{v})\,\mathrm{d}V \qquad \frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\,\vec{v}) = -\vec{v}\cdot\nabla\rho - \rho\,\nabla\cdot\vec{v}$$

continuity Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\,V) = 0 \qquad \frac{1}{\rho V}\frac{\mathrm{d}}{\mathrm{d}t}(\rho\,V) = \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}V}{\mathrm{d}t} = V\,\nabla\cdot\vec{v} \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\,\nabla\cdot\vec{v}$$

momentum

$$\rho\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\nabla P + \rho\,\vec{g} + \ldots \qquad \rho\,\frac{\partial\vec{v}}{\partial t} + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla P + \rho\,\vec{g} + \ldots$$

Lorentz $(\nabla\wedge\vec{B})\wedge\vec{B}/4\pi$ Coriolis $2\,\vec{\Omega}\wedge\vec{v}$ differential rotation $(\vec{v}\cdot\nabla\vec{\Omega})\wedge\vec{v}$ viscosity $\mu\nabla^2\vec{v}$

Poisson

$$\vec{g} = -\nabla\Phi \qquad \nabla^2\Phi = 4\pi\,G\,\rho \qquad \Phi(\vec{r}) = -G\int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}V$$

Energy (adiabatic)

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\gamma P}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

SMALL CHANGES

Stix section 5.2.1

linear perturbations

$P = P_0 + P_1 \qquad \rho = \rho_0 + \rho_1 \qquad \vec{v} = \vec{v}_0 + \vec{v}_1 = \vec{v}_1 \qquad P_1 \ll P_0 \qquad \rho_1 \ll \rho_0 \qquad \vec{v} \ll c_s$

Lagrangian perturbations (S 5.15 with displacement $\vec{\delta r} = \boldsymbol{\xi}$)

$$\delta P = P_1 + \vec{\delta r} \cdot \nabla P_0 \qquad \rho = \rho_1 + \vec{\delta r} \cdot \nabla \rho_0 \qquad \vec{v} = \frac{\partial \vec{\delta r}}{\partial t}$$

continuity (S 5.13)

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \, \vec{v}) = 0 \qquad \rho_1 + \nabla \cdot (\rho_0 \, \vec{\delta r}) = 0$$

momentum (S 5.14)

$$\rho_0 \frac{\partial^2 \vec{\delta r}}{\partial t^2} = \rho_0 \, \frac{\partial \vec{v}}{\partial t} = -\nabla P_1 + \rho_0 \, \vec{g_1} + \rho_1 \, \vec{g_0} = -\nabla P_1 - \rho_0 \, \nabla \Phi_1 + \frac{\rho_1}{\rho_0} \nabla P_0$$

Cowling approximation (S 5.2.3, 5.29): waves ⇒ many radial sign changes ⇒ average out

$$\nabla^2 \Phi_1 = 4\pi G \rho_1 \qquad \Phi_1 = -G \int \frac{\rho_1(r')}{|r - r'|} \, \mathrm{d}r' \approx 0$$

adiabatic energy (S 5.10)

$$\frac{P_1}{P_0} = \gamma \, \frac{\rho_1}{\rho_0} \qquad \frac{\delta P}{P_0} = \gamma \, \frac{\delta \rho}{\rho_0}$$

ACOUSTIC WAVES IN HOMOGENEOUS MEDIUM

momentum, continuity, energy equations without gravity or mean-state derivatives

$$\rho_0 \frac{\partial^2 \delta\vec{r}}{\partial t^2} = -\nabla P_1 \qquad \frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \vec{v_1} \qquad P_1 = \gamma \frac{\gamma P_0}{\rho_0} \rho_1$$

with sound speed $c_{\rm s}^2 \equiv \gamma P_0/\rho_0$ Mach $M \equiv V/c_{\rm s}$ ideal gas $c_{\rm s}^2 = \gamma\, k\, T/\mu \sim T/\mu$

$$\frac{\partial^2 P_1}{\partial t^2} = c_{\rm s}^2 \left(-\rho_0 \nabla \cdot \frac{\partial \vec{v_1}}{\partial t} \right) = c_{\rm s}^2 \nabla^2 P_1$$

$$\frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \nabla \cdot \frac{\partial \vec{v_1}}{\partial t} = c_{\rm s}^2 \nabla^2 \rho_1$$

$$\frac{\partial^2 \vec{v_1}}{\partial t^2} = -\frac{1}{\rho_0} \nabla \cdot \left(c_{\rm s}^2 \frac{\partial \rho_1}{\partial t} \right) = c_{\rm s}^2 \nabla^2 \vec{v_1}$$

space-time variable separation

$$P(x,y,z,t) \equiv g(x,y,z)\ f(t) \Rightarrow \frac{1}{f}\frac{\partial^2 f}{\partial t^2} = \frac{c_{\rm s}^2}{g} \nabla^2 g \equiv -\omega^2 \Rightarrow \frac{\partial^2 f}{\partial t^2} + \omega^2 f = 0 \Rightarrow f(t) = a_t\, {\rm e}^{i\omega t} + b_t\, {\rm e}^{-i\omega t}$$

spatial variable separation $g(x,y,z) = g_x(x)\ g_y(y)\ g_z(z)$

$$\nabla^2 g + \frac{\omega^2}{c_{\rm s}^2} g = 0 \Rightarrow \frac{\partial^2 g_x}{\partial x^2} + k_x^2\, g_x = 0 \Rightarrow g_x(x) = a_x\ {\rm e}^{ik_x x} + b_x\ {\rm e}^{-ik_x x} \text{ etc.}$$

plane waves

$$P_1 = P_0\, \gamma\, M\ {\rm e}^{i(\vec{k}\cdot\vec{r} \pm i\omega t)} \qquad \rho_1 = \rho_0\, M\ {\rm e}^{i(\vec{k}\cdot\vec{r} \pm i\omega t)} \qquad \vec{v_1} = c_{\rm s} M\ {\rm e}^{i(\vec{k}\cdot\vec{r} \pm i\omega t)}$$

spatial wavenumber $\vec{k} = (k_x, k_y, k_z) \parallel \vec{v}$ dispersion relation $k^2 = |\vec{k}|^2 = \omega^2/c_{\rm s}^2$

PLANE WAVE PROPERTIES

waves in homogeneous gaseous medium

$$\frac{P_1}{P_0\gamma M} = \frac{\rho_1}{\rho_0 M} = \frac{\vec{v_1}}{c_\mathrm{s} M} = \mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{k} \parallel \vec{v_1} \parallel \delta\vec{r} \quad \text{longitudinal}$$

wavelength, wavenumber, period, (angular) frequency

full cycle at $kx = 2\pi \Rightarrow x = \lambda \qquad k_x = 2\pi/\lambda$

full cycle at $\omega t = 2\pi \Rightarrow t = P \qquad \omega = 2\pi/P = 2\pi f$

plane wave: same phase across plane with $\vec{k}\cdot\vec{r} = 0$ so $\perp \vec{k}$ and $\perp \vec{v}$

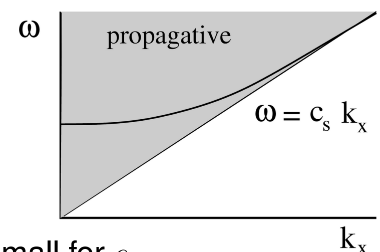
phase velocity = propagation wave pattern

wave $\vec{v} = (\dot{x}, 0, 0)$: equal phase: $k_x x - \omega t = c \qquad x = (\omega/k_x)\, t + c \qquad v_\mathrm{phase} = \omega/k = c_s$

group velocity = propagation envelope non-monochromatic wave train $L_x\,\Delta k_x \approx 1/2$

$$\mathrm{e}^{i[(\vec{k}+\Delta\vec{k})\cdot\vec{r}-(\omega+\Delta\omega)t]} = \mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega t)}\;\mathrm{e}^{i(\Delta\vec{k}\cdot\vec{r}-\Delta\omega\, t)} \Rightarrow v_\mathrm{group} = \frac{\partial\omega}{\partial k}$$

dispersion relation $|\vec{k}|^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2/c_\mathrm{s}^2$ and diagnostic diagram



shaded: real solutions = propagating waves

boundary line: $k_y = k_z = 0$, waves only in x

smaller k_x towards upper left

ω axis: $k_x = 0$, waves only in y and/or z

curve: $\omega = \sqrt{c_\mathrm{s}^2 k_x^2 + c}$ for constant $k_y^2 + k_z^2 = c$

long periods: only at large wavelength $\lambda_x = 2\pi/k_x$

non-shaded: period P too long for k_x, wavelength λ_x too small for c_s

imaginary solutions $k_y^2 < 0$ or $k_z^2 < 0$: exponential growth/decay in y or z

VERTICAL ACOUSTIC WAVES IN ISOTHERMAL ATMOSPHERE

“atmosphere” $\equiv$ gas layer plane-parallel in x and y, stratified by gravity in z

$$\nabla T_0 = 0 \qquad \nabla P_0 = \rho_0 \vec{g} \Rightarrow \frac{\partial P_0}{\partial z} = -\rho_0 g = -P_0 \frac{\gamma g}{c_\mathrm{s}^2} \equiv -\frac{P_0}{H} \qquad H = \frac{c_\mathrm{s}^2}{\gamma g} = \frac{RT}{\mu g} \qquad k_h^2 \equiv k_x^2 + k_y^2$$

linearized conservation equations with $\vec{v} = (v_h, v_z)$ and $\vec{g} = (0, g)$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \vec{v} - \vec{v} \cdot \nabla \rho_0 \qquad \rho_0 \frac{\partial \vec{v}}{\partial t} = \rho_1 \vec{g} - \nabla P_1 \qquad \frac{\partial P_1}{\partial t} + \vec{v} \cdot \nabla P_0 = -\gamma P_0 \nabla \cdot \vec{v}$$

give inhomogeneous wave equations for P_1, ρ_1, $\vec{v}$

$$\frac{\partial^2 \vec{v}}{\partial t^2} = c_\mathrm{s}^2 \nabla^2 \vec{v} - (\gamma - 1) \vec{g} \nabla \cdot \vec{v} - \nabla(\vec{v} \cdot \vec{g})$$

vertical motion $\vec{v} = (0, v_z)$

$$\frac{\partial^2 v_z}{\partial t^2} = c_\mathrm{s}^2 \nabla^2 v_z - \gamma g \frac{\partial v_z}{\partial z} \qquad \text{try} \quad v_z = \mathrm{e}^{i(k_z z - \omega t)}$$

$$\omega^2 = c_\mathrm{s}^2 k_z^2 + i \gamma g k_z \qquad k_z = -i \frac{\gamma g}{2 c_\mathrm{s}^2} \pm \frac{1}{c_\mathrm{s}} \sqrt{\omega^2 - \omega_\mathrm{a}^2} \qquad \omega_\mathrm{a} \equiv \frac{\gamma g}{2 c_\mathrm{s}}$$

$$v_z = \mathrm{e}^{(\gamma g / 2 c_\mathrm{s}^2) z} \, \mathrm{e}^{i \left[\pm (z/c_\mathrm{s}) \sqrt{\omega^2 - \omega_\mathrm{a}^2} - \omega t \right]} = \mathrm{e}^{z/2H} \, \mathrm{e}^{i \left[\pm (z/c_\mathrm{s}) \sqrt{\omega^2 - \omega_\mathrm{a}^2} - \omega t \right]}$$

properties

- amplitude $\sim \mathrm{e}^{z/2H} \sim \sqrt{\rho_o(z)}$ = energy conservation $(1/2)\rho_0 v_z^2$ up to non-linear regime
- $\omega > \omega_\mathrm{a}$ propagating plane wave
- $\omega < \omega_\mathrm{a}$ slow perturbation = evanescent wave: whole atmosphere up and down in phase
  aperiodic growth or decay with z depending on upper or lower piston

SLANTED AG WAVES IN ISOTHERMAL ATMOSPHERE 1

Stix section 5.2.4 *Hines 1960CaJPh..38.1441H* *Whitaker 1963ApJ...137..914W*

substitute into linearised conservation laws

$$\frac{P_1}{P_0 \mathcal{P}} = \frac{\rho_1}{\rho_0 \mathcal{R}} = \frac{v_h}{\mathcal{X}} = \frac{v_z}{\mathcal{Z}} = \mathrm{e}^{i(\boldsymbol{k}_h \cdot x + \boldsymbol{k}_z \cdot z - \omega\, t)}$$

complex $\mathcal{P}, \mathcal{R}, \mathcal{X}, \mathcal{Z}, \boldsymbol{k}_h, \boldsymbol{k}_z$

separation of vertical-only solution: $\Im k_z = \gamma g / 2 c_\mathrm{s}^2$

$$\frac{P_1}{P_0 \mathcal{P}} = \frac{\rho_1}{\rho_0 \mathcal{R}} = \frac{v_h}{\mathcal{X}} = \frac{v_z}{\mathcal{Z}} = \mathrm{e}^{z/2H}\ \mathrm{e}^{i(k_h x + k_z z - \omega\, t)}$$

real k_h, k_z

polarisation relations

$$\mathcal{P} = \gamma\, \omega^2 \left[k_z - i\, (1 - \frac{\gamma}{2})\, \frac{g}{c_\mathrm{s}^2} \right] \qquad \mathcal{R} = \omega^2\, k_z + i\, (\gamma - 1)\, g\, k_h^2 - i\, \frac{\gamma}{2}\, \frac{g\, \omega^2}{c_\mathrm{s}^2}$$

$$\mathcal{X} = \omega\, k_h\, c_\mathrm{s}^2 \left[k_z - i\, (1 - \frac{\gamma}{2})\, \frac{g}{c_\mathrm{s}^2} \right] \qquad \mathcal{Z} = \omega\, (\omega^2 - k_h^2\, c_\mathrm{s}^2)$$

amplitude & phase

$$\mathcal{X} \equiv |\mathcal{X}|\, \mathrm{e}^{i\phi} \qquad v_h = |\mathcal{X}|\, \mathrm{e}^{z/2H}\ \mathrm{e}^{i(k_h x + k_z z - \omega\, t + \phi)} \qquad \phi = \arctan(\Im X / \Re X)$$

dispersion relation

$$\omega^4 - \omega^2\, c_\mathrm{s}^2\, (k_h^2 + k_z^2) + (\gamma - 1)\, g^2\, k_h^2 - \frac{\gamma^2 g^2 \omega^2}{4 c_\mathrm{s}^2} = 0$$

$\omega_\mathrm{a} \equiv \dfrac{\gamma g}{2 c_\mathrm{s}}$ $\quad \omega_\mathrm{g} \equiv \dfrac{g}{c_\mathrm{s}} \sqrt{\gamma - 1}$ $\quad \omega_\mathrm{g} < \omega_\mathrm{a}$ since $\gamma < 2$ $\quad \gamma = 5/3 \Rightarrow \omega_\mathrm{g} = 0.98\, \omega_\mathrm{a}$

$$(\omega^2 - \omega_\mathrm{a}^2)\, \frac{\omega^2}{c_\mathrm{s}^2} - \omega^2\, (k_h^2 + k_z^2) + \omega_\mathrm{g}^2\, k_h^2 = 0$$

SLANTED AG WAVES IN ISOTHERMAL ATMOSPHERE 2

reordered dispersion relations

$$k_h^2 = \frac{(\omega_\mathrm{a}^2 - \omega^2)\,(\omega^2/c_\mathrm{s}^2) + \omega^2\,k_z^2}{\omega_\mathrm{g}^2 - \omega^2} \qquad k_z^2 = (\omega_\mathrm{g}^2 - \omega^2)\,\frac{k_h^2}{\omega^2} - (\omega_a^2 - \omega^2)/c_\mathrm{s}^2$$

$$\omega^2 = \frac{1}{2}\left[\omega_\mathrm{a}^2 + c_\mathrm{s}^2\,k^2 \pm \sqrt{(\omega_\mathrm{a}^2 + c_\mathrm{s}^2\,k^2)^2 - 4\,c_\mathrm{s}^2\,k_h^2\,\omega_\mathrm{g}^2}\;\right]$$

Brunt-Väisälä frequency Schwarzschild convective instability criterion: $N^2 < 0$

$$\omega_\mathrm{g} \equiv \frac{g}{c_\mathrm{s}}\sqrt{\gamma - 1} \qquad N_\mathrm{BV}^2 \equiv \frac{g^2}{c_\mathrm{s}^2}(\gamma - 1) + \frac{g}{T}\frac{\mathrm{d}T}{\mathrm{d}z}$$

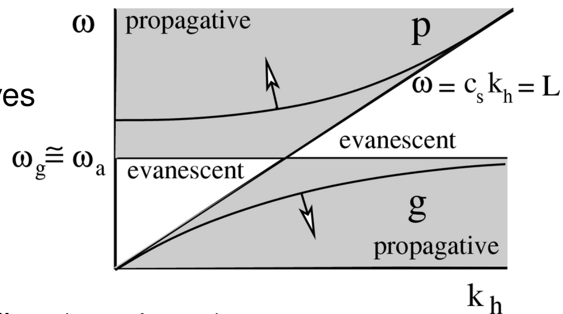
isothermal $\omega_\mathrm{g} = N_\mathrm{BV}$ $\quad \gamma = 5/3 : \omega_\mathrm{g} = 0.98\,\omega_\mathrm{a}$

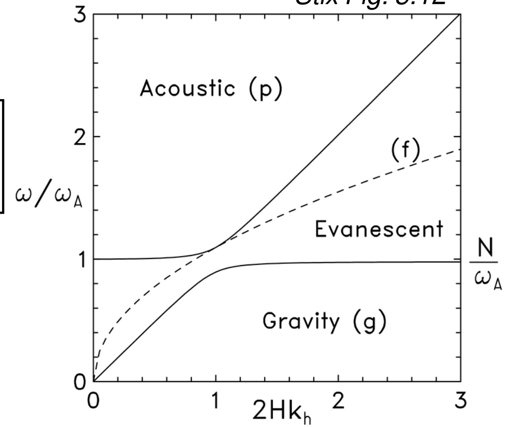
with $L^2 \equiv c_\mathrm{s}^2\,k_h^2$ and $\omega_\mathrm{g} \approx \omega_\mathrm{a}$

$$k_z^2 = \frac{1}{\omega^2\,c_\mathrm{s}^2}\left[(\omega_\mathrm{g}^2 - \omega^2)\,k_h^2\,c_\mathrm{s}^2 - (\omega_\mathrm{a}^2 - \omega^2)\,\omega^2\right] \approx \frac{1}{\omega^2\,c_\mathrm{s}^2}(\omega^2 - \omega_{\mathrm{g}\approx\mathrm{a}}^2)\,(\omega^2 - L^2)$$

diagnostic diagram

- shaded: two ω roots per (k_h, k_z) = propagative waves
  - $\omega > \omega_\mathrm{a}$ and $\omega > L$: acoustic wave
  - $\omega < \omega_\mathrm{g}$ and $\omega < L$: internal gravity wave
  - Cowling: p (pressure) or g (gravity)
- blank: $k_z^2 < 0$: evanescent in z
- line $\omega^2 = c_\mathrm{s}^2\,k_h^2 = L^2$: horizontal "Lamb" waves
- line $\omega^2 = (\omega_\mathrm{g}/\omega_\mathrm{a})^2\,c_\mathrm{s}^2\,k_h^2$: gravity-wave cutoff at small k_h (not shown)
- upper curve: constant k_z, hyperbolic, $\omega^2 - \omega_\mathrm{a}^2 \approx c_\mathrm{s}^2 k_z^2$ for small k_h
- lower curve: constant k_z, hyperbolic, $\omega_\mathrm{g}^2 - \omega^2 \approx k_z^2/k_h^2$ for large k_h
- arrows: increasing k_z





phase speed $\theta \equiv$ angle between $\vec{k}$ and k_z

$$v_{\rm phase}^2 = \frac{\omega^2}{k^2} = \frac{1}{2}\left[\frac{\omega_{\rm a}^2}{k^2} + c_{\rm s}^2 \pm \sqrt{\left(\frac{\omega_{\rm a}^2}{k^2} + c_{\rm s}^2\right)^2 - 4\,c_{\rm s}^2 \sin^2\theta\,\frac{\omega_{\rm g}^2}{k^2}}\,\right]$$

$+\sqrt{\ldots} = p$ waves, $-\sqrt{\ldots} = g$ waves

p waves: $v_{\rm phase} \geq c_{\rm s}$ $\quad v_{\rm phase} \downarrow$ for $\omega \uparrow$

g waves: $0 \leq v_{\rm phase} \leq (\omega_g/\omega_a)\,c_{\rm s}$ $\quad v_{\rm phase} \uparrow$ for $\theta \to \pi/2$

$v_{\rm phase} = 0$ for $\theta = 0, \pi$

group velocity

$$v_{\rm group}^{\rm hor} = \frac{c_{\rm s}^2\,(\omega^2 - \omega_{\rm g}^2)\,k_h}{\omega\,(2\,\omega^2 - \omega_{\rm a}^2 - c_{\rm s}^2\,k^2)} \qquad v_{\rm group}^{\rm ver} = \frac{c_{\rm s}^2\,\omega^2\,k_z}{\omega\,(2\,\omega^2 - \omega_{\rm a}^2 - c_{\rm s}^2\,k^2)}$$

pure p regime $\omega \gg \omega_{\rm a}$

$\omega^2 = c_{\rm s}^2\,k^2 \quad v_{\rm phase} \approx c_{\rm s} \quad v_{\rm group}^{\rm hor,ver} \approx (c_{\rm s}^2/\omega)\,k_{h,v} \leq c_{\rm s} \quad P_1,\, \rho_1,\, v_h,\, v_z$ in phase

pure g regime $k_h^2 \gg \omega_{\rm g}^2/c_{\rm s}^2$

dispersion: $\omega^2 = (k_h^2/k^2)\,\omega_g^2 = \sin^2\theta\,\omega_{\rm g}^2$

non-vertical only (“blob” concept needs horizontal info)

$v_{\rm group}^{\rm hor} \sim k_h,\, v_{\rm group}^{\rm ver} \sim -k_z$ slanted upward waves transport energy downward

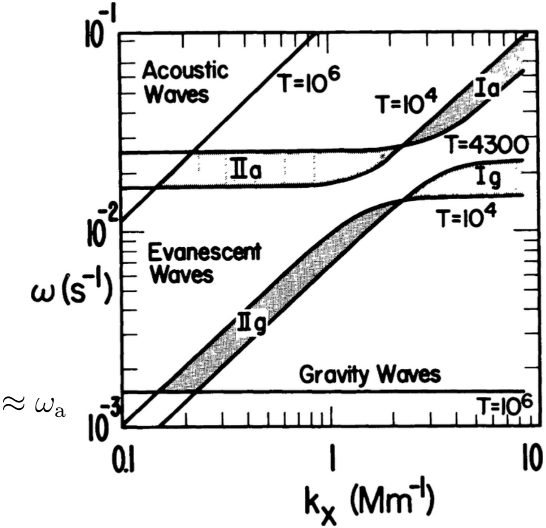
evanescent regime

k_z imaginary, k_h free $\quad \mathcal{P}, \mathcal{R}$ imaginary, $\mathcal{Z}$ real: P_1 and $\vec{v_z}$ 90 deg out of phase

non-radial fundamental mode (f) $\sim$ ocean wind wave: $\omega = \sqrt{g\,k_h}$

WAVE TRAPPING

Stein & Leibacher 1974ARA&A..12..407S



temperature sensitivities

sound speed: $c_{\rm s} \equiv \sqrt{\gamma \dfrac{P_0}{\rho_0}} = \sqrt{\gamma \dfrac{RT}{\mu}} \sim \sqrt{T}$

acoustic cutoff frequency: $\omega = \omega_{\rm a} \equiv \dfrac{\gamma\, g}{2\, c_{\rm s}} \sim 1/\sqrt{T}$

horizontal acoustic-wave line (Lamb): $\omega = c_{\rm s}\, k_h$

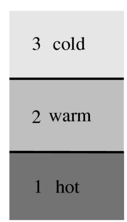
gravity cutoff line: $\omega = (\omega_{\rm g}/\omega_{\rm a})\, c_{\rm s}\, k_h$

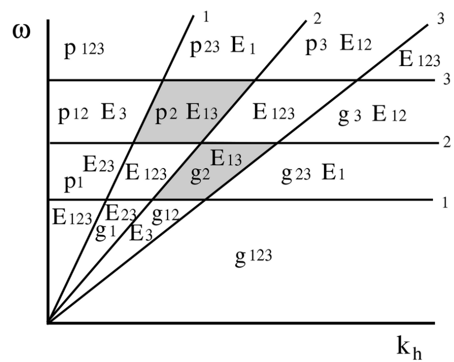
horizontal gravity-wave frequency: $\omega = N_{\rm BV} \approx \omega_{\rm g} \approx \omega_{\rm a}$

wave trapping

refraction at increasing wave speed (wavefront bending)
turn-around where waves become horizontal (node in k_z)
anti-node reflection at cutoff (infinite phase speed)
p-mode cavity walls: $\omega_{\rm a}$ cutoff and Lamb-line refraction
g-mode cavity walls: gravity cutoff line and $N_{\rm BV}$ refraction

example: three temperature regimes
shaded: 2 = cavity p and g modes





SOLAR CAVITIES

p: refractive turnaround at Lamb line

$$\omega^2 = c_s^2\, k_h^2 = c_s^2\, \frac{l(l+1)}{r^2} \sim \frac{T\, l^2}{r^2}$$

lowest l modes probe deepest

g: refractive turnaround at Brunt-Väisälä N^2

no g waves in convection zone

interior g modes? lift whole envelope?

atmospheric g waves?

near solar center

Lamb reflection $\omega^2 \sim 1/r^2$

ω_a and $N_{BV} \sim g \sim r \Rightarrow N^2 \sim r^2$

only radial $p(l\!=\!0)$ modes reach center

five-minute oscillation ($\omega \approx 0.02$ Hz)

upper reflection = ω_a (here N_{AC})

interior reflection = Lamb turnaround

($\approx$ closed/open ends of my flute)

p modes evanescent where detected

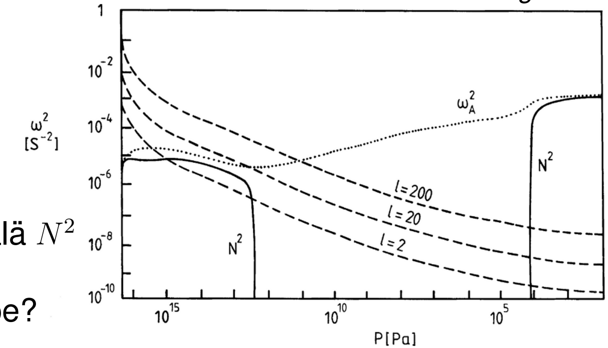
three-minute oscillation ($\omega \approx 0.035$ Hz)

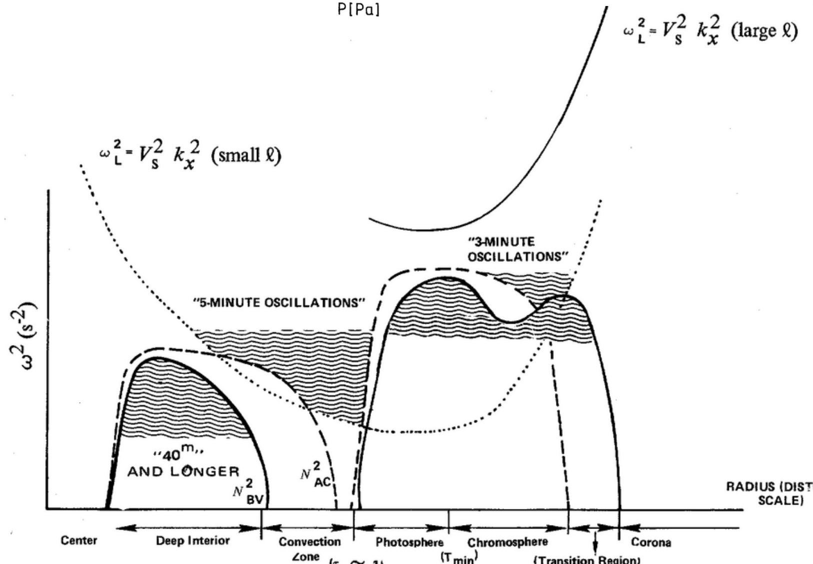
here: cavity from coronal T rise

but: T rise too warped

reflection/mode conversion at $\vec{B}$

Stix Fig. 5.13





Leibacher & Stein *1981suas.nasa..263L*

Ando & Osaki 1975PASJ...27..581A

(a) *Basic Equations and Boundary Conditions*

The basic equations governing nonadiabatic radial pulsations in the radiative atmosphere were formulated by UNNO (1965), in which the radiative transfer was treated in the Eddington approximation. UNNO and SPIEGEL (1966) have demonstrated that the Eddington approximation in the radiative heat equation is very useful in three-dimensional time-dependent problems. We thus utilize this in the present paper. The basic equations governing nonadiabatic nonradial oscillations in the radiative atmosphere are then given as

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})=0\,, \tag{1}$$

$$\frac{d\boldsymbol{v}}{dt}=-\frac{1}{\rho}\boldsymbol{\nabla}P-\boldsymbol{g}\,, \tag{2}$$

$$c_P\rho\left(\frac{dT}{dt}-\nabla_{\rm ad}\frac{T}{P}\frac{dP}{dt}\right)=-\pi\boldsymbol{\nabla}\cdot\boldsymbol{F}\,, \tag{3}$$

$$\boldsymbol{F}=-\frac{4}{3\kappa\rho}\boldsymbol{\nabla}J\,, \tag{4}$$

and

$$J=\frac{ac}{4\pi}T^4+\frac{c_P}{4\pi\kappa}\left(\frac{dT}{dt}-\nabla_{\rm ad}\frac{T}{P}\frac{dP}{dt}\right), \tag{5}$$

THE SUN IS A SPHERE

Wolff *1972ApJ...177L..87W*

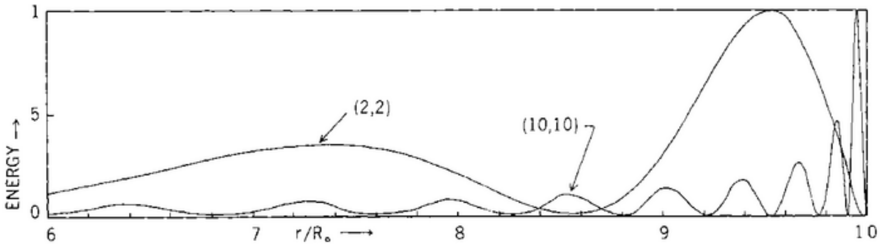
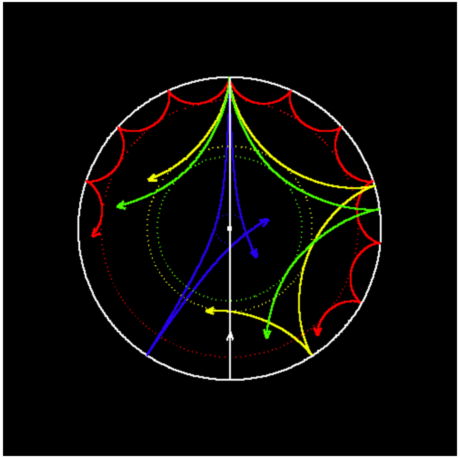
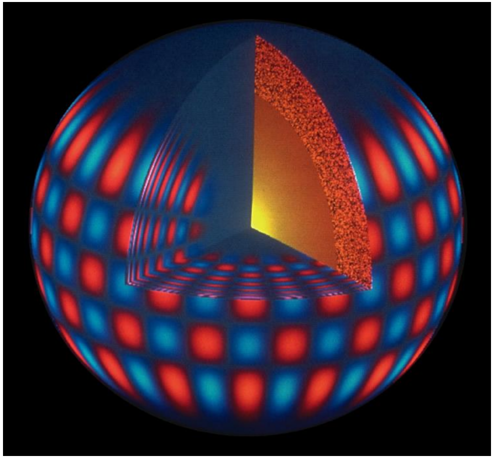


FIG. 1.—Radial distribution of kinetic energy (arbitrary units) in spherical shells of unit thickness for two nonradial *p*-modes of solar oscillation.





NUMERICAL p-MODE PREDICTION

Ando & Osaki 1975PASJ...27..581A

We assume as usual that the small perturbation of a physical quantity f is written as

$$\left.\begin{matrix} f'(r,\theta,\varphi,t) \\ \delta f(r,\theta,\varphi,t) \end{matrix}\right\} = \left.\begin{matrix} f'(r) \\ \delta f(r) \end{matrix}\right\} Y_l^m(\theta,\varphi) e^{i\sigma t} , \tag{6}$$

where (r,θ,φ) is the spherical polar coordinates, the Eulerian and the Lagrangian perturbations are denoted by prime ($'$) and δ, and $Y_l^m(\theta,\varphi)$ is the spherical harmonics. We introduce a nondimensional frequency ω and five nondimensional variables x, p, θ, j, and λ defined by

$$\left.\begin{aligned} &\omega^2=(R^3/GM)\sigma^2 , \\ &x=\delta r/r , \quad p=P'/\rho g r , \quad \theta=\delta T/T , \\ &j=\delta J/J , \quad \lambda=\delta L_r/L_s , \end{aligned}\right\} \tag{7}$$

where $L_r=4\pi^2 r^2 F$ is the luminosity at radius r and L_s is the luminosity at the surface. By linearizing equations (1)-(5), we then obtain four first-order linear differential equations and one auxiliary equation:

POWER RIDGES $f-l$ (aka $k-\omega$) DIAGRAM

Stix section 5.2

assume constant temperature gradient with depth $d=-z$

$$\frac{\mathrm{d}T}{\mathrm{d}d} \approx C = \left(\frac{\mathrm{d}T}{\mathrm{d}d}\right)_{\mathrm{ad}} \qquad T(d) = \left(\frac{\mathrm{d}T}{\mathrm{d}d}\right)_{\mathrm{ad}} d \qquad -\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}d}\right)_{\mathrm{ad}} = \frac{(\gamma-1)\,g}{c_{\mathrm{s}}^2} \qquad c_{\mathrm{s}}^2 \approx (\gamma-1)\,g\,d$$

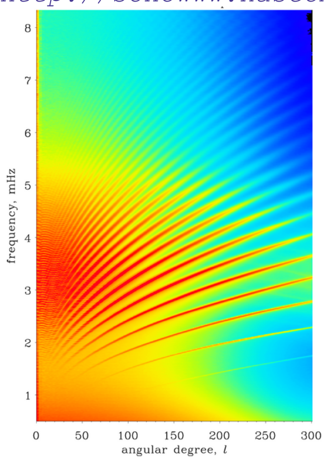
lower reflection occurs at depth $d=\delta$ where $\omega^2 = c_{\mathrm{s}}^2\,k_h^2$:

$$\delta = \frac{c_{\mathrm{s}}^2}{(\gamma-1)\,g} = \frac{\omega^2}{(\gamma-1)\,g\,k_h^2}$$

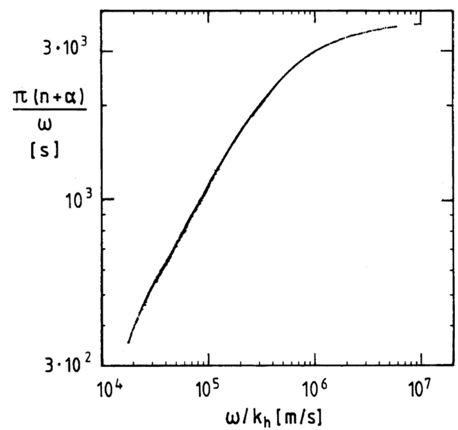
standing wave has $(n+1/2)\,\pi = \int k_z\,\mathrm{d}d = \omega\int_{d=\delta}^{d=0}\frac{1}{c_{\mathrm{s}}}\,\mathrm{d}d = \frac{2\omega\sqrt{\delta}}{\sqrt{(\gamma-1)\,g}} = \frac{2\,\omega^2}{(\gamma-1)\,g\,k_h}$

parabolas $\omega_n^2 = \frac{1}{2}\pi\,(n+1/2)\,(\gamma-1)\,g\,k_h \qquad k_h^2 = l\,(l+1)/R_\odot^2$

http://sohowww.nascom.nasa.gov



Stix Fig. 5.16



Duvall's law with $\alpha = 1.58$

FIVE-MINUTE OSCILLATION: THEORY

Robert J. Rutten
http://www.staff.science.uu.nl/~rutte101

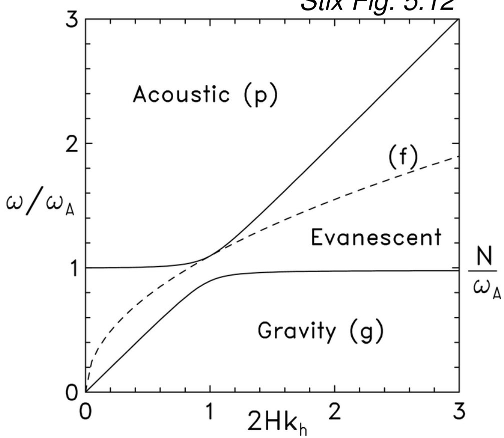
Literature

Jørgen Christensen-Dalsgaard: “Stellar Oscillations”, 2003
http://astro.phys.au.dk/~jcd/oscilnotes

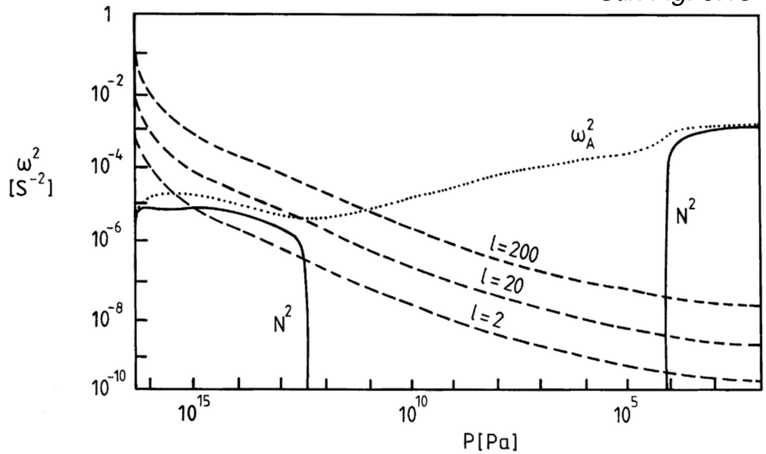
Michael Stix: “The Sun”, 2004, second edition, Springer

Rob Rutten: “Fotosferische snelheidsvelden”, 1983
http://www.staff.science.uu.nl/~rutte101/Collegedictaat_Fotosferisch.html

Stix Fig. 5.12



Stix Fig. 5.13



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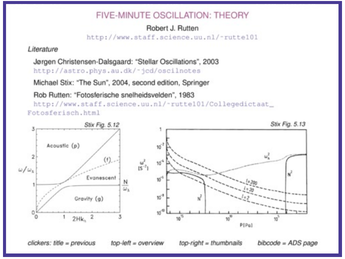
FIVE-MINUTE OSCILLATION: THEORY
Robert J. Rutten

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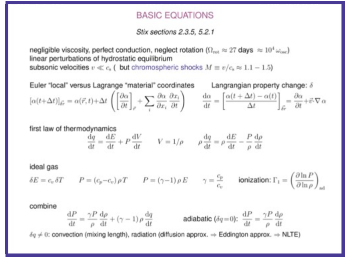
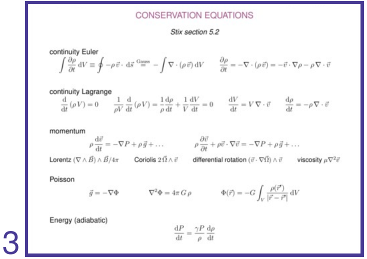
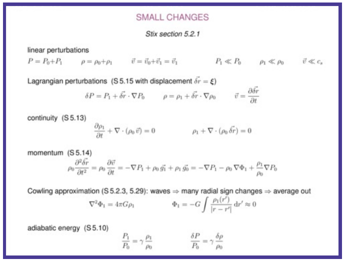
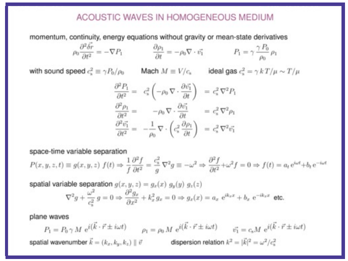
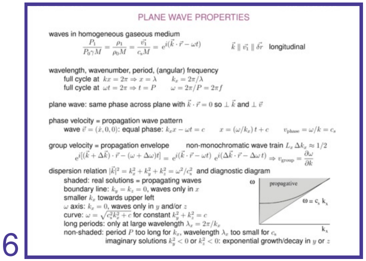
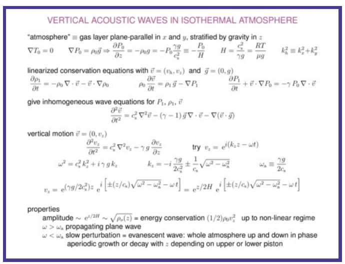

BASIC EQUATIONS
Stix sections 2.3.5, 5.2.1

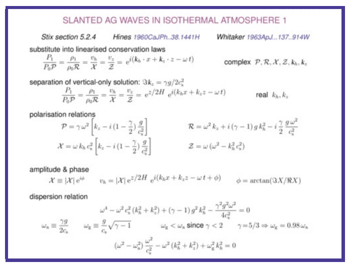
3

CONSERVATION EQUATIONS
Stix section 5.2

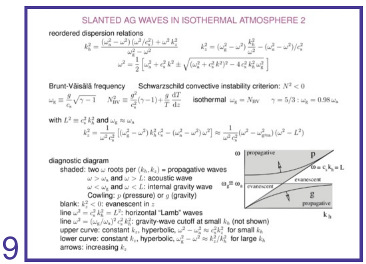
4

SMALL CHANGES
Stix section 5.2.1

5

ACOUSTIC WAVES IN HOMOGENEOUS MEDIUM

6

PLANE WAVE PROPERTIES

7

VERTICAL ACOUSTIC WAVES IN ISOTHERMAL ATMOSPHERE

8

SLANTED AG WAVES IN ISOTHERMAL ATMOSPHERE 1

9

SLANTED AG WAVES IN ISOTHERMAL ATMOSPHERE 2

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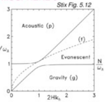
SLANTED AG WAVES IN ISOTHERMAL ATMOSPHERE 3

Stix Fig. 5.12



phase speed $\theta \equiv$ angle between $\vec{k}$ and k_z

$$v^2_{\rm phase} = \frac{\omega^2}{k^2} = \frac{1}{2}\left[\frac{\omega_a^2}{k^2} + c_s^2 \pm \sqrt{\left(\frac{\omega_a^2}{k^2}+c_s^2\right)^2 - 4\,c_s^2 \sin^2\theta\,\frac{\omega_g^2}{k^2}}\right]$$

$+\sqrt{\ldots} = p$ waves, $-\sqrt{\ldots} = g$ waves

p waves: $v_{\rm phase} \geq c_s$ $\quad v_{\rm phase} \downarrow$ for $\omega \uparrow$

g waves: $0 \leq v_{\rm phase} \leq (\omega_g/\omega_a)\,c_s$ $\quad v_{\rm phase} \uparrow$ for $\theta \to \pi/2$

$v_{\rm phase} = 0$ for $\theta = 0, \pi$

group velocity

$$v^{\rm hor}_{\rm group} = \frac{c_s^2\,(\omega^2-\omega_g^2)\,k_h}{\omega\,(2\,\omega^2-\omega_a^2-c_s^2\,k^2)} \qquad v^{\rm ver}_{\rm group} = \frac{c_s^2\,\omega^2\,k_z}{\omega\,(2\,\omega^2-\omega_a^2-c_s^2\,k^2)}$$

pure p regime $\omega \gg \omega_a$

$\omega^2 = c_s^2\,k^2$ $\quad v_{\rm phase} \approx c_s$ $\quad v^{\rm hor,ver}_{\rm group} \approx (c_s^2/\omega)\,k_{h,v} \leq c_s$ $\quad P_1, \rho_1, v_h, v_z$ in phase

pure g regime $k_h^2 \gg \omega_g^2/c_s^2$

dispersion: $\omega^2 = (k_h^2/k^2)\,\omega_g^2 = \sin^2\theta\,\omega_g^2$

non-vertical only ("blob" concept needs horizontal info)

$v^{\rm hor}_{\rm group} \sim k_h$, $v^{\rm ver}_{\rm group} \sim -k_z$ slanted upward waves transport energy downward

evanescent regime

k_z imaginary, k_h free $\quad \mathcal{P}, \mathcal{R}$ imaginary, $\mathcal{Z}$ real: P_1 and $\vec{v_z}$ 90 deg out of phase

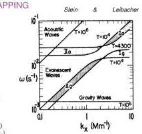
non-radial fundamental mode (f) $\sim$ ocean wind wave: $\omega = \sqrt{g\,k_h}$

11

WAVE TRAPPING

1974ARA&A..12..407S

Stein & Leibacher



temperature sensitivities

sound speed: $c_s \equiv \sqrt{\gamma\frac{P_0}{\rho_0}} = \sqrt{\gamma\frac{RT}{\mu}} \sim \sqrt{T}$

acoustic cutoff frequency: $\omega = \omega_a \equiv \frac{\gamma\,g}{2\,c_s} \sim 1/\sqrt{T}$

horizontal acoustic-wave line (Lamb): $\omega = c_s\,k_h$

gravity cutoff line: $\omega = (\omega_g/\omega_a)\,c_s\,k_h$

horizontal gravity-wave frequency: $\omega = N_{\rm BV} \approx \omega_g \approx \omega_a$

wave trapping

refraction at increasing wave speed (wavefront bending)

turn-around where waves become horizontal (node in k_z)

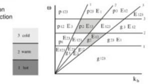
anti-node reflection at cutoff (infinite phase speed)

p-mode cavity walls: ω_a cutoff and Lamb-line refraction

g-mode cavity walls: gravity cutoff line and $N_{\rm BV}$ refraction

example: three temperature regimes

shaded: 2 = cavity p and g modes



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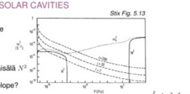
SOLAR CAVITIES

Stix Fig. 5.13



p: refractive turnaround at Lamb line

$\omega^2 = c_s^2\,k_h^2 = c_s^2\,\frac{l(l+1)}{r^2} \sim \frac{T\,l^2}{r^2}$

lowest l modes probe deepest

g: refractive turnaround at Brunt-Väisälä N^2

no g waves in convection zone

interior g modes? lift whole envelope?

atmospheric g waves?

near solar center

Lamb reflection $\omega^2 \sim 1/r^2$

ω_a and $N_{\rm BV} \sim g \sim r \Rightarrow N^2 \sim r^2$

only radial $p(l\!=\!0)$ modes reach center

five-minute oscillation ($\omega \approx 0.02$ Hz)

upper reflection = ω_a (here $N_{\rm AC}$)

interior reflection = Lamb turnaround

($\approx$ closed/open ends of my flute)

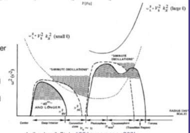
p modes evanescent where detected

three-minute oscillation ($\omega \approx 0.035$ Hz)

here: cavity from coronal T rise

but: T rise too warped

reflection/mode conversion at $\vec{B}$



Leibacher & Stein 1981suas.nasa..263L

13

NUMERICAL p-MODE PREDICTION

Ando & Osaki 1975PASJ...27..581A

(a) *Basic Equations and Boundary Conditions*

The basic equations governing nonadiabatic radial pulsations in the radiative atmosphere were formulated by UNNO (1965), in which the radiative transfer was treated in the Eddington approximation. UNNO and SPIEGEL (1966) have demonstrated that the Eddington approximation in the radiative heat equation is very useful in three-dimensional time-dependent problems. We thus utilize this in the present paper. The basic equations governing nonadiabatic nonradial oscillations in the radiative atmosphere are then given as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \tag{1}$$

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\nabla P - \boldsymbol{g}\,, \tag{2}$$

$$c_P\rho\left(\frac{dT}{dt} - \nabla_{\rm ad}\frac{T}{P}\frac{dP}{dt}\right) = -\kappa\nabla\cdot\boldsymbol{F}\,, \tag{3}$$

$$\boldsymbol{F} = -\frac{4}{3\kappa\rho}\nabla J\,, \tag{4}$$

and

$$J = \frac{ac}{4\pi}T^4 + \frac{c_P}{4\pi\kappa}\left(\frac{dT}{dt} - \nabla_{\rm ad}\frac{T}{P}\frac{dP}{dt}\right), \tag{5}$$

14

THE SUN IS A SPHERE

Wolff 1972ApJ...177L..87W

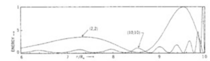
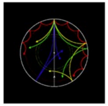
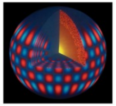


FIG. 1.—Radial distribution of kinetic energy (arbitrary units) in spherical shells of unit thickness for two nonradial p-modes of solar oscillation.





15

NUMERICAL p-MODE PREDICTION

Ando & Osaki 1975PASJ...27..581A

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where (r,θ,φ) is the spherical polar coordinates, the Eulerian and the Lagrangian perturbations are denoted by prime ($'$) and δ, and $Y_l^m(\theta,\varphi)$ is the spherical harmonics. We introduce a nondimensional frequency ω and five nondimensional variables x, p, θ, j, and λ defined by

$$\left.\begin{array}{l} \omega^2 = (R^3/GM)\sigma^2\,, \\ x = \delta r/r\,,\quad p = P'/\rho g r\,,\quad \theta = \delta T/T\,, \\ j = \delta J/J\,,\quad \lambda = \delta L_r/L_s\,, \end{array}\right\} \tag{7}$$

where $L_r = 4\pi^2 r^2 F$ is the luminosity at radius r and L_s is the luminosity at the surface. By linearizing equations (1)-(5), we then obtain four first-order linear differential equations and one auxiliary equation:

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POWER RIDGES $f-l$ (aka $k-\omega$) DIAGRAM

Stix section 5.2

assume constant temperature gradient with depth $d = -z$

$$\frac{{\rm d}T}{{\rm d}d} \approx C = \left(\frac{{\rm d}T}{{\rm d}d}\right)_{\rm ad} \qquad T(d) = \left(\frac{{\rm d}T}{{\rm d}d}\right)_{\rm ad} d \qquad -\frac{1}{T}\left(\frac{{\rm d}T}{{\rm d}d}\right)_{\rm ad} = \frac{(\gamma-1)\,g}{c_s^2} \qquad c_s^2 \approx (\gamma-1)\,g\,d$$

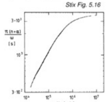
lower reflection occurs at depth $d = \delta$ where $\omega^2 = c_s^2\,k_h^2$: $\quad \delta = \frac{c_s^2}{(\gamma-1)\,g} = \frac{\omega^2}{(\gamma-1)\,g\,k_h^2}$

standing wave has $(n+1/2)\,\pi = \int k_z\,{\rm d}d = \omega\int_{d=\delta}^{d=0}\frac{1}{c_s}\,{\rm d}d = \frac{2\omega\sqrt{\delta}}{\sqrt{(\gamma-1)\,g}} = \frac{2\,\omega^2}{(\gamma-1)\,g\,k_h}$

parabolas $\omega_n^2 = \frac{1}{2}\pi\,(n+1/2)\,(\gamma-1)\,g\,k_h \qquad k_h^2 = l\,(l+1)/R_\odot^2$

http://sohowww.nascom.nasa.gov

Stix Fig. 5.16



Duvall's law with $\alpha = 1.58$

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FIVE-MINUTE OSCILLATION: THEORY

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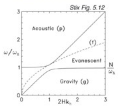
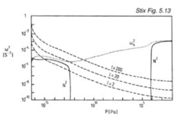
Stix Fig. 5.12



Stix Fig. 5.13



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