Problem set 1 for Student Seminar Theor. Physics (ns-tp501m)

Problems are due to at Wed Sep 24 2008. (28 points in total.)

1 The geodesic equation. *(4 points)*

By making use of the action principle, derive the geodesic equation for the 4-velocity of a point particle, $u^\mu = dx^\mu/d\tau$ from the following general relativistic action for a point particle,

$$S_{\text{point particle}} = -mc\int ds = -mc\int d\tau\Big(g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big)^{1/2} . \qquad (1)$$

In proving this you may use $\nabla_\alpha g_{\mu\nu} = 0$ (see Problem 2c below). What is the special relativistic limit of the action (1)?

2. General covariance and tensors. *(8 points)*

(a) *(2 points)*

Show that

$$\int d^4x\sqrt{-g} \qquad (2)$$

represents a generally covariant measure. The symbol $g = \det[g_{\alpha\beta}]$ denotes the determinant of the metric tensor. You may find the following definition of the determinant of a 2-indexed tensor $t_{\mu\nu}$ useful,

$$\epsilon_{\mu\nu\rho\sigma}\det[t_{\alpha\beta}] = \epsilon_{\alpha\beta\gamma\delta}t_{\mu\alpha}t_{\nu\beta}t_{\rho\gamma}t_{\sigma\delta} , \qquad (3)$$

where $\epsilon_{\mu\nu\rho\sigma}$ represents a totally antisymmetric Levi-Cività ϵ-symbol in 3+1 dimensions, such that $\epsilon_{0123} = 1$, and it is antisymmetric under the exchange of any two indices. The ϵ-symbol vanishes whenever any two indices are identical.

Show that $\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}$ transforms as a four-indexed covariant tensor, while $\epsilon^{\mu\nu\rho\sigma}/\sqrt{-g}$ transforms as a contravariant tensor.

(b) *(2 points)*

Show that the covariant derivative of a two indexed contravariant tensor field $T^{\mu\nu}$ reads

$$\nabla_\alpha T^{\mu\nu} \equiv T^{\mu\nu}{}_{;\alpha} = T^{\mu\nu}{}_{,\alpha} + \Gamma^\mu_{\rho\alpha}T^{\rho\nu} + \Gamma^\nu_{\rho\alpha}T^{\mu\rho} , \qquad (4)$$

where $_{,\alpha} \equiv \partial/\partial x^\alpha$.

Hint: Make use of *e.g.* Eq. (39) in Part I of lecture notes,

$$B_{\alpha\beta;\gamma} = B_{\alpha\beta,\gamma} - \Gamma^\mu_{\alpha\gamma}B_{\mu\beta} - \Gamma^\mu_{\beta\gamma}B_{\mu\alpha} . \qquad (5)$$

(c) *(2 points)*

Show that the covariant derivative of the metric tensor vanishes (assume the Levi-Cività/Christoffel connection),

$$\nabla_\alpha g_{\mu\nu} = 0\,. \tag{6}$$

(d) *(2 points)*

Show that the Einstein curvature tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - (1/2)\mathcal{R}g_{\mu\nu}$ satisfies the following Bianchi identity

$$\nabla^\nu G_{\mu\nu} = 0\,, \tag{7}$$

where $\mathcal{R}_{\mu\nu}$ denotes the Ricci curvature tensor and $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ is the Ricci curvature scalar.

Hint: Show first that the following cyclic derivative property for the Riemann curvature tensor,

$$\nabla_\gamma \mathcal{R}_{\mu\nu\alpha\beta} + \nabla_\alpha \mathcal{R}_{\mu\nu\beta\gamma} + \nabla_\beta \mathcal{R}_{\mu\nu\gamma\alpha} = 0\,, \tag{8}$$

and then contract the appropriate indices.

3. Thermal sphere. *(9 points)*

Newtonian spherically symmetric gravitating systems of many particles (all of an identical mass m) satisfy the Poisson equation for the gravitational Newton potential ϕ_N,

$$\nabla^2 \phi_N = 4\pi G_N \rho_N\,, \tag{9}$$

where ρ_N denotes the mass density, which in a spherically symmetric system is a function of the distance r from the center of mass, $\phi_N = \phi_N(r)$. In an equilibrated system, the distribution of particles can be approximated by the thermal distribution function $f = f(r, v)$, which is a function of $v = |\vec{v}|$ and $r = |\vec{r}|$ only,

$$f = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\Big(-\frac{v^2/2 + \phi_N}{\sigma^2}\Big)\,, \tag{10}$$

where v is particle's velocity, $\phi_N = \phi_N(r)$ Newton's gravitational potential, $\sigma^2 = \langle \vec{v}^2 \rangle / 3 \equiv k_B T/m$ and

$$\rho_N(r) = \int d^3 v f\,. \tag{11}$$

(a) *(2 points)*

Show that ϕ_N satisfies the following (mean field) equation of motion,

$$\frac{d^2}{dr^2}\phi_N + \frac{2}{r}\frac{d}{dr}\phi_N = 4\pi G_N \rho_1 \exp\Big(-\frac{\phi_N(r)}{\sigma^2}\Big)\,. \tag{12}$$

(b) *(3 points)*

Show that the following *Ansatz* solves the Poisson equation (9) (see also Eq. (12))

$$\begin{aligned} \rho_N(r) &= \frac{\sigma^2}{2\pi G_N r^2} \\ \phi_N(r) &= -\sigma^2 \ln\left(\frac{\sigma^2}{2\pi G_N \rho_1 r^2}\right). \end{aligned} \tag{13}$$

This solution is known as the *thermal sphere*, and it is the only known analytic solution of Eq. (12).

Next, show that the mass inside radius r reads,

$$M(r) = \frac{2k_B T}{m G_N} r\,. \tag{14}$$

Discuss the limits $r \to 0$ and $r \to \infty$.

(c) *(4 points) Light deflection.*

Calculate the deflection angle of light in the presence of a mass distribution of a thermal sphere, given by the thermal sphere potential (13) by making use of the formula,

$$\vec{\alpha}(d) = -\frac{2}{c^2}\int d\ell\, \nabla_\perp \phi_N(\vec{x}) \tag{15}$$

where $\nabla_\perp$ is the gradient operator in the lens plane, whose two components are transversal (perpendicular) to the photon path, ℓ is the distance along the light geodesic, and d is the impact parameter, which is the shortest distance from the center of mass ($\vec{x} = 0$) to the geodesic. Assume that the photon path can be approximated by an almost straight line, $\vec{x} \simeq (d, 0, z)$, ($z \in \{-\infty, \infty\}$). In this case Eq. (15) reduces to,

$$\alpha(d) = -\frac{2}{c^2}\int_{-\infty}^{\infty} dz \partial_x \phi_N(\vec{x})\,. \tag{16}$$

Assume that the mass distribution of an elliptical galaxy can be well approximated by a thermal sphere, with a typical dispersion of a velocity component $\sigma = 300$ km/s. Calculate the light deflection angle originating at a distant point source (quasar or galaxy). Express your answer in arc seconds. Comment on the dependence of the deflection angle α on d.

4. *Fermions in curved space-times.* (*7 points*)

Consider the following covariant Dirac action for fermions in curved space times,

$$S_{\text{fermion}} = \int d^4x \sqrt{-g}\Big(\bar{\psi} i \gamma^\mu \nabla_\mu \psi - m_\psi \bar{\psi}\psi\Big)\,, \tag{17}$$

where $\bar{\psi} = \psi^\dagger \gamma^0(x)$ and m_ψ denotes the fermion mass. The covariant derivative acting on a fermion field is given in terms of the spin connection Γ_μ as,

$$\nabla_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi \,, \tag{18}$$

which is in turn defined by

$$\nabla_\mu \gamma_\nu \equiv \partial_\mu \gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0 \,. \tag{19}$$

By recalling that,

$$\gamma_\mu = e^b_\mu \gamma_b \,, \tag{20}$$

one can show that a solution of (19) is

$$\Gamma_\mu = -\frac{1}{8} e^\nu_c \Big(\partial_\mu e_{\nu d} - \Gamma^\alpha_{\mu\nu} e_{\alpha d} \Big) [\gamma^c, \gamma^d] \,, \tag{21}$$

where $[\gamma^c, \gamma^d] = \gamma^c\gamma^d - \gamma^d\gamma^c$ denotes the commutator. Here we are using $a, b, c, d, ..$ for the tangent space indices, on which γ^a and η_{ab} are space-time independent, and $\mu, \nu, \rho, \sigma, ..$ for the space-time indices.

We consider homogeneous conformal space-times, with the conformally flat metric tensor,

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \equiv e^b_\mu(x) e^c_\nu(x) \eta_{bc} \,, \qquad \eta_{bc} = \mathrm{diag}(1, -1, -1, -1) \,, \quad (b, c = 0, 1, 2, 3) \,, \tag{22}$$

where the scale factor $a = a(\eta)$ is a function of conformal time η (*e.g.* in de Sitter space-time, $a = -1/(H_I \eta)$ ($\eta < 0$), where H_I denotes the Hubble parameter of de Sitter space).

(a) (*1 point)*

Check that in conformal space-times the vierbein has the form

$$e^c_\mu(x) = \delta^c_\mu a(x) \,, \qquad e^\mu_c(x) = \delta^\mu_c a(x)^{-1} \,, \qquad (\mu = 0, 1, 2, 3;\ c = 0, 1, 2, 3) \,. \tag{23}$$

and that the Dirac matrices obeying the correct anticommutation relation are,

$$\gamma^\mu(x) = e^\mu_b \gamma^b = a(x)^{-1} \delta^\mu_b \gamma^b \,. \tag{24}$$

(b) (*3 points)*

Show that the Levi-Cività connection $\Gamma^\alpha_{\mu\nu}$ and the spin connection Γ_μ are of the form,

$$\begin{aligned} \Gamma^\alpha_{\mu\nu} &= \frac{a'}{a}\Big(\delta^0_\mu \delta^\alpha_\nu + \delta^0_\nu \delta^\alpha_\mu - \delta^\alpha_0 \eta_{\mu\nu} \Big) \\ \Gamma_\mu &= \frac{1}{4}\frac{a'}{a}\Big[\gamma^0, \gamma^b\Big]\eta_{\mu b} \,, \end{aligned} \tag{25}$$

where $a' = da/d\eta$ and $\eta_{\mu b} = \mathrm{diag}(1, -1, -1, -1)$.

(c) *(3 points)*

Show that, when contracted with gamma matrices, the covariant derivative acting on a spinor acquires the form,

$$i\nabla\!\!\!/ \equiv e_c^{\mu} i\gamma^c \nabla_\mu = a^{-\frac{5}{2}} i\gamma^c \partial_c a^{\frac{3}{2}} \,. \tag{26}$$

Show that from here it then follows that the Dirac equation for fermions in homogeneous conformal space-times can be written as

$$\Big(i\gamma^c\partial_c - a m_\psi\Big)\psi_{cf} = 0\,, \qquad \psi_{cf} = a^{3/2}\psi\,, \tag{27}$$

where ψ_{cf} denotes a conformally rescaled spinor. Comment on the physical implications of this result.