Gauge-invariant regularization of Yang-Mills fields

by

G. 't Hooft

Institute for Theoretical Physics
University of Utrecht
Utrecht, the Netherlands

Abstract

A procedure is described which leads to finite, gauge invariant expressions for the S-matrix in gauge field models. It is based on the replacement of the number N, denoting the number of dimensions of Minkowsky space, by a continuous parameter N. It is shown that the amplitudes obey the Ward identities for gauge fields on the one hand, and the Cutkosky rules and causality on the other hand. The method is found to be applicable to diagrams with an arbitrary number of closed loops.

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1. Introduction

Interest in gauge fields has revived recently, since the Feynman rules appear to be of a renormalizable type [1,2,3]. A spontaneous symmetry-breaking can be introduced, arising from a translational shift of certain field quantities[4,5]. The Yang-Mills bosons then acquire a mass, and even this case appears to be renormalizable, despite of the vector character of the particles[6,7]. A field theory of this type has been proposed to describe weak interactions between leptons[8,9,10].

Still, the fact that divergencies cancel is not enough for a theory to be renormalizable. It must also be shown that renormalization can be carried out in such a way that the Ward identities are not violated as a consequence of certain boundary effects, or by some finite contribution of gauge-non invariant regulators. We have seen that this is presicely what happens in the P.C.A.C. case: it is impossible to regularize a fermion triangle graph, expressing π^0 decay, in such a way that P.C.A.C. and electromagnetic gauge invariance are both satisfied. In the case of Yang-Mills fields the Ward identities are a necessary condition for unitarity. So it is worthwhile to check whether such anomalies are absent in the case of gauge fields.

Now, in ref. 3) it was found how to regularize diagrams with one closed loop in a gauge invariant way: a fifth component was assigned to all Lorentz-indices inside the closed loop, and thus we obtained massive regulators. The propagator identities used to prove Ward identities, also hold in a five-dimensional space, and so the Ward identities are obeyed by these regularized amplitudes.

The purpose of this article is to show how to extend this method of introducing more dimensions in Minkowsky space, for diagrams with more closed loops. We write down, formally, the contribution of a certain graph as a function of the number of dimensions, N. Only for a limited number of values for N this integral has a well-defined meaning, so at this stage analytic continuation towards non-integer values of N is impossible. Nevertheless, we can <u>define</u> an analytic function of N, called "pseudo-integral", which represents the integral of the graph as soon as it converges, for integer values of N. For those integer values of N where the integral diverges, our function may exhibit poles. This definition is made unique by requiring also that a number of algebraic manipulations can be carried out with these integrals. For instance, integration and differentiation must commute, and shifts of the integration variables may not alter the integral. This property is the key with which we are able to prove the validity of the Ward identities for these pseudo-integrals.

Finally, we want to take the limit $N \rightarrow 4$. But, as soon as the usual integral for $N = 4$ diverges, a pole occurs at $N = 4$. The residue of this pole is a polynomial in terms of the external momenta. Now we know that the amplitude is gauge-invariant for all N, so also the residues of the pole are gauge-invariant. So we must be able to cancel this pole by means of a gauge-invariant counterterm in the Lagrangian:

$$\Delta \mathcal{L} = \frac{-C}{N-4} \mathcal{L}_{YM} \qquad (1.1)$$

where C represents the residue of the pole.
Now the limit $N \to 4$ exists and yields the desired expression.

In section 3 it is shown that the regularization scheme we propose is equivalent with ordinary Pauli-Villars regularization, after addition of further local counterterms in the Lagrangian. Finally, an illustrative example is worked out in the appendix: the regularization procedure is demonstrated for the photon self-energy in quantum electrodynamics.

2. Algebraic properties of pseudo-integrals

Suppose we have an N-dimensional space V_N with Minkowsky metric: $g_{\mu\upsilon} = (1,\ldots 1,\ -1)$. Then we have

$$\int_{V_N} d^N p \left[(p-k)^2 + m^2 - i\epsilon \right]^{-\alpha} = i\,\pi^{\frac{1}{2}N}\, m^{N-2\alpha}\, \frac{\Gamma(\alpha - \frac{1}{2}N)}{\Gamma(\alpha)}\,, \qquad (2.1)$$

as long as the integral converges.
Now we define the "pseudo-integral" over p, for any non-integer value of N, of the function $[(p-k)^2 + m^2 - i\epsilon]^{-\alpha}$ by the same expression (2.1).

One may verify that the pseudo-integral has the following properties:

(i) Differentiation with respect to m^2, and pseudo-integration, commute.

(ii) The pseudo-integral is independent of the origin of integration. So pseudo-integration and differentiation with respect to k also commute.

(iii) If for integer values of N the integral converges then the pseudo-integral equals the integral, even if it is a linear combination of divergent parts.*

(iv) If α is an integer, then the pseudo-integral may have poles at those integer values of N for which the ordinary integral diverges. Note that it is impossible to define other pseudo-integrals with properties (i), (ii), (iii), without such poles.

From eq. (2.1) we derive

$$\int d^N p \, \frac{[(p-k)^2]^{\beta}}{[(p-k)^2 + m^2 - i\epsilon]^{\alpha}} = i\pi^{\frac{1}{2}N} m^{N+2\beta-2\alpha} \frac{\Gamma(\beta+\frac{1}{2}N)\Gamma(\alpha-\beta-\frac{1}{2}N)}{\Gamma(\frac{1}{2}N)\,\Gamma(\alpha)} \,. \tag{2.2}$$

By differentiation of eq. (2.1) with respect to k_μ we derive

$$\int d^N p \, \frac{(p-k)_\mu}{[(p-k)^2 + m^2 - i\epsilon]^{\alpha}} = 0, \tag{2.3a}$$

and

$$\int d^N p \, \frac{(p-k)_\mu (p-k)_\upsilon}{[(p-k)^2 + m^2 - i\epsilon]^{\alpha}} = \frac{\delta_{\mu\upsilon}}{N} \int d^N p \, \frac{(p-k)^2}{[(p-k)^2 + m^2 - i\epsilon]^{\alpha}} \,. \tag{2.3b}$$

The left hand side of eq. (2.3) could also be calculated by means of symmetric integration. So, we conclude that the definition (2.1) for pseudo-integrals is only consistent if

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* Note that this is only true if the integrals converge strictly. In all other cases the replacement $N \leftrightarrow 4$ may lead to non-vanishing contributions due to the pole in the Γ-functions.

we have

$$\sum_{\mu} \delta_{\mu\mu} = N \; , \tag{2.4}$$

also for non-integer N.

Let us now consider the Feynman rules given in refs. [1,2,3,6], and also the auxiliary vertices (4.3a,b,c) of ref. [3]. An arbitrary diagram can always be written in terms of functions like in eqs. (2.1), (2.2), (2.3) by using Feynman's auxiliary variables:

$$\frac{1}{a_1 \cdots\cdots a_n} = (n-1)! \int_0^1 \cdots \int_0^1 dx_1 \cdots dx_n \; \frac{\delta(\Sigma x - 1)}{[a_1 x_1 + \; \cdots \; + a_n x_n]^n} \; . \tag{2.5}$$

Now we can give a well defined prescription how to calculate the amplitude for non-integer N using the pseudo-integrals (extension of these to two or more Minkowsky-variables is straightforward). Whenever the sum of Lorentz-indices occurs we must insert eq.(2.4). Now, because we may freely shift integration variables, the formal proof of the Ward identities given in ref. [3] may be applied here, so that these identities are satisfied for all N.

Next, let us consider the limit $N \to 4$. Because some of the integrals diverge, the pseudo-integrals will exhibit poles for $N = 4$.

If all Feynman variables x_i are non-zero, then the residues of these poles must be contact terms, i.e. polynomials of a certain degree in terms of the external momenta. (This can easily be proven by differentiating with respect to these momenta until the integrals converge, using properties (i) and (ii). According to property (iii) the poles then disappear). Now the only possible gauge invariant contact term of this degree is of the form of the Lagrangian from which we started. So we add to the Lagrangian a counter term $\Delta\mathcal{L}$ which cancels this pole. If we proceed this way, order by order in perturbation theory, then all poles vanish, and we have a counter term of the following type:

$$\Delta\mathcal{L} = \frac{g^2\mathcal{L}_1}{N-4} + \frac{g^4\mathcal{L}_2}{N-4} + \dots \tag{2.6}$$

where $\mathcal{L}_{1,2,\dots}$ are all gauge invariant, and have dimension 4 or less.

Now the limit $N \to 4$ exists, and is gauge invariant (i.e. satisfies all Ward identities) by construction.

3. Comparison with Pauli-Villars regularization

Now we show that the S-matrix for gauge gields defined this way is unitary. Let us recall the proof of ref. [3]. The Ward identities are satisfied, so only the cutting rule remains to be proven. It is possible to show[11)] that this cutting rule can be extended to non-integer N, thus ensuring its validity for $N \to 4$, provided we have removed all poles by the appropriate counter terms. Here we shall prove the cutting rule by showing that our regularization method is in fact equivalent with Pauli-Villars regularization[12)] up to local counter terms in the Lagrangian. Suppose this equivalency has been shown for all diagrams with no more than $n-1$ closed loops, and suppose that we have added the necessary local counter terms to the Pauli-Villars regulirized graphs. Now, consider an irreducible diagram with n closed loops. Our aim is to show that our calculation with pseudo-integrals for this graph equals the regularized expression, up to a contact term, and terms that vanish for large regulator masses. The proof is simple. Differentiate both expressions with respect to the external momenta until the superficial divergence (i.e. divergence obatained by power counting) of the obtained integrals is less than zero. The contact term must then have disappeared. Now we make use of the following lemma, which holds for renormalized field theories:

lemma. If the superficial divergency of a graph is less than zero, then at least one of the internal lines is not contained in any divergent subgraph.

So, if we take the momentum of that line as the last integration variable, then this last integration converges. The regulator of that line does not contribute on the one hand, and the integral equals the pseudo-integral on the other hand. The contribution of the lower-order subgraphs to the integrand is equal for both cases, by assumption. This completes our proof by induction.

4. Fermions

We conclude that if the field variables and the Lagrangian can be written down in a space with any number N of dimensions, then the system can be renormalized, up to arbitrary order of the coupling constant. The Ward identities for the renormalized amplitudes are not in conflict with each other.

Fermions can also be included, because what we need for proving a Ward identity is only the commutation properties of the γ - matrices:

$$\gamma_\mu \gamma_\upsilon + \gamma_\upsilon \gamma_\mu = \delta_{\mu\upsilon} \ . \tag{4.1}$$

We now set up a consistent algebra for these γ - matrices. This algebra consists in defining the trace of an arbitrary sequence of γ - matrices, in such a way that eq. (4.1) holds. For instance:

$$\begin{aligned} \mathrm{Tr}\ 1 &\equiv 4, \\ \mathrm{Tr}\ \gamma_\mu \gamma_\upsilon &\equiv 4\,\delta_{\mu\upsilon}\ , \\ \mathrm{Tr}\ \gamma_\mu\gamma_\upsilon\gamma_\kappa\gamma_\lambda &\equiv 4(\delta_{\mu\upsilon}\delta_{\kappa\lambda} - \delta_{\mu\kappa}\delta_{\upsilon\lambda} + \delta_{\mu\lambda}\delta_{\upsilon\kappa})\ , \\ &\text{etc.} \end{aligned} \tag{4.2}$$

Eq. (4.2) can be continued by considering a spinor representation of an N-dimensional Minkowsky space*). Together with eq. (2.4) we now have a well-defined pseudo-integrand, and we may continue as in sect. 3. Again, the Ward identities for fermion pseudo-integrals can be derived using combinatorics, and eq. (4.1).

One thing must be kept in mind: γ_5 being a pseudoscalar in four dimensions, must be considered as a four-index tensor if $N \neq 4$. This however would be untenable if for instance,

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*) Note that an overall factor in eq. (4.2) is only fixed by the unitarity requirement in 4 dimensions. Thus eq. (4.2) may be altered by a factor f(N) with f(4) = 1.

gauge transformations of the kind $\psi' = e^{i\Lambda\gamma_5}\psi$ are contained in the gauge group. This is why for instance Weinberg's model of weak interactions is not renormalizable this way [8,10]. Difficulties of this type also underly the well-known anomalies of Adler's triangle graph [13,14].

The author wishes to thank prof. M. Veltman, who proved the validity of cutting rules in this formalism.

Appendix

It is illustrative to show how our regularization method works in the case of the vacuum polarization in quantum electrodynamics. The pseudo-integral is

$$\Pi_{\mu\upsilon}(k,N) = \frac{-ie^2}{(2\pi)^4} \int d^N p \; \frac{\mathrm{Tr}(m - i\gamma p)\gamma_\mu (m - i\gamma(p+k))\gamma_\upsilon}{(p^2 + m^2)((p+k)^2 + m^2)} =$$

$$\frac{-ie^2}{(2\pi)^4} \int d^N p \int_0^1 dx \; \frac{\mathrm{Tr}(m - i\gamma p)\gamma_\mu (m - i\gamma(p+k))\gamma_\upsilon}{[(p+xk)^2 + m^2 + x(1-x)k^2]^2} \; . \tag{A.1}$$

We may shift $p + xk \rightarrow p$ and already insert eq. (2.3a). The trace is calculated following eqs. (4.2):

$$\Pi_{\mu\upsilon}(k,N) =$$

$$\frac{-4\,ie^2}{(2\pi)^4} \int d^N p \int_0^1 dx \; \frac{m^2\delta_{\mu\upsilon} + x(1-x)[2k_\mu k_\upsilon - k^2\delta_{\mu\upsilon}] - 2p_\mu p_\upsilon + p^2\delta_{\mu\upsilon}}{[p^2 + m^2 + x(1-x)k^2]^2} \tag{A.2}$$

According to eq. (2.3b) we replace

$$p_\mu p_\upsilon \rightarrow \frac{1}{N} \, p^2 \, \delta_{\mu\upsilon} \; . \tag{A.3}$$

Now eq. (2.2) is applied, and the result is

$$\Pi_{\mu\upsilon}(k,N) = \frac{-8\,e^2}{(2\pi)^4} \pi^{\frac{1}{2}N} \Gamma(2 - \tfrac{1}{2}N) \int_0^1 dx \; x(1-x)$$

$$\left(m^2 + x(1-x)k^2\right)^{\frac{1}{2}N - 2} [k^2\delta_{\mu\upsilon} - k_\mu k_\upsilon] . \tag{A.4}$$

Note that this expression satisfies the Ward identity

$$k_\mu \, \Pi_{\mu\upsilon}(k,N) = 0 \; . \tag{A.5}$$

Now for $N \to 4$ we have

$$\Gamma\,(2-\tfrac{1}{2}N) \to \frac{-2}{N-4}\,. \tag{A.6}$$

So, indeed, there is a pole. Its residue is

$$\frac{16e^2\pi^2}{(2\pi)^4}\int_0^1 dx\; x(1-x)\;[k^2\delta_{\mu\upsilon} - k_\mu k_\upsilon]\,. \tag{A.7}$$

This is a gauge invariant contact term. So, let us add the following second order counter term to the Lagrangian,

$$\Delta\mathcal{L} = (\,\tfrac{1}{4}\,F_{\mu\upsilon}F_{\mu\upsilon})\,\frac{8\,e^2\pi^2}{(2\pi)^4}\;\Gamma(2-\tfrac{1}{2}N)\int_0^1 x(1-x)\;dx, \tag{A.8}$$

which gives rise to another contribution to the self-energy:

$$\Delta\,\Pi_{\mu\upsilon}(k,N) = \frac{8\,e^2\pi^2}{(2\pi)^4}\,\Gamma(2-\tfrac{1}{2}N)\int_0^1 dx\; x(1-x)[k^2\delta_{\mu\upsilon} - k_\mu k_\upsilon]\,. \tag{A.9}$$

Inserting eq. (A.6) and

$$\lim_{N\to 4}\,\frac{1}{N-4}\,\left(z^{\frac{1}{2}N-2} - 1\right) = \tfrac{1}{2}\log z, \tag{A.10}$$

we find the regularized self-energy

$$\begin{aligned}\Pi^{reg}_{\mu\upsilon}(k) &\equiv \lim_{N\to 4}\,\Big(\Pi_{\mu\upsilon}(k,N) + \Delta\,\Pi_{\mu\upsilon}(k,N)\Big)\\ &= \frac{8\,e^2\pi^2}{(2\pi)^4}\int_0^1 dx\; x(1-x)\left[\log\left(1+\frac{x(1-x)k^2}{m^2}\right) + \log \pi m^2\right]\left[k^2\delta_{\mu\upsilon} - k_\mu k_\upsilon\right]\end{aligned} \tag{A.11}$$

The term $\log\pi\, m^2$ may be eliminated by another finite, gauge-invariant counter term in the Lagrangian.

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