Exercise sheet 2: 10 Mar

NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 20 Mar, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **Energy-momentum conservation.** Show that the energy momentum conservation $D^\mu T_{\mu\nu} = 0$ follows from the Bianchi identity and Einstein's equations.

2. **Newtonian limit.** Let's consider the Newtonian limit in which the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $h_{\mu\nu}$ is viewed as infinitesimal with respect to the Minkowski metric.

(a) Show that the Christoffel symbols, the Ricci tensor and the Ricci scalar to first order in h are given by

$$\begin{aligned} \Gamma^\alpha_{\mu\nu} &= \frac{1}{2}\eta^{\alpha\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \\ R_{\mu\nu} &= \frac{1}{2}(-\partial^2 h_{\mu\nu} + \partial_\alpha\partial_\mu h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\mu\partial_\nu h^\alpha_\alpha) \\ R &= -\partial^2 h^\mu_\mu + \partial^\mu\partial^\nu h_{\mu\nu}. \end{aligned} \tag{2}$$

(b) Let's assume from now on the metric to be stationary, $\partial_t h_{\mu\nu} = 0$. Show that the geodesic equation for a slowly moving particle (i.e. $\dot{x}^\mu \approx (1, \dot{x}^i)$) reduces to

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}. \tag{3}$$

Therefore we can identify $-\frac{1}{2}h_{00}$ with the Newtonian gravitational potential $\Phi(x)$.

(c) Show that $h_{\mu\nu} = 2\Phi\delta_{\mu\nu}$ solves the Einstein equations with $T_{00}(x) = -\rho(x)$ (and the other components zero) when Φ satisfies the Newton's gravity law

$$\partial^2\Phi(x) = 4\pi G\rho(x). \tag{4}$$

(d) The components T_{ii} describe the pressure of the source and if they are non-zero they will contribute to the gravitational field. However, show that when we integrate T_{ii} over the region containing the source, the contribution averages to zero,

$$\int d^3x T_{ii} = 0, \quad i = 1, 2, 3. \tag{5}$$

Hint: calculate $\partial_1 \int dx^2 dx^3 T_{11}(x)$ using the energy-momentum conservation which in our static case reads $\partial^i T_{ij} = 0$.

3. **Deriving the Schwarzschild solution.** Any 4-dimensional spherically symmetric static metric can be written in the form

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{6}$$

where $A(r)$ and $B(r)$ are functions of r. We will therefore take (6) as ansatz to find a solution to the Einstein equations.

(a) What should the asymptotics ($r \to \infty$) of $A(r)$ and $B(r)$ be for the metric to look like Minkowski space for large r?

(b) Show that the non-vanishing Christoffel symbols are (up to symmetry in the lower indices)

$$\begin{array}{lllll} \Gamma^r_{tt} = \frac{A'}{2B} & \Gamma^t_{rt} = \frac{A'}{2A} & \Gamma^r_{rr} = \frac{B'}{2B} & \Gamma^r_{\theta\theta} = -\frac{r}{B} & \Gamma^\theta_{r\theta} = \frac{1}{r} \\ \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{B} & \Gamma^\phi_{r\phi} = \frac{1}{r} & \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^\phi_{\theta\phi} = \cot\theta & \end{array} \tag{7}$$

(c) Show that the non-vanishing components of the Ricci tensor are

$$\begin{aligned} R_{tt} &= \frac{1}{2B}\left(A'' - \frac{A'B'}{2B} - \frac{(A')^2}{2A} + \frac{2A'}{r}\right), \\ R_{rr} &= \frac{1}{2A}\left(-A'' + \frac{(A')^2}{2A} + \frac{A'B'}{2B} + \frac{2AB'}{rB}\right), \\ R_{\theta\theta} &= 1 - \frac{1}{B}\left(1 + \frac{rA'}{2A} - \frac{rB'}{2B}\right), \end{aligned} \tag{8}$$

and $R_{\phi\phi}$ (but you do not need to calculate $R_{\phi\phi}$).

(d) Applying the vacuum Einstein equations $R_{\mu\nu} = 0$, we get an (overdetermined) system of differential equations. Show that a solution must satisfy $(AB)' = 0$ and $(r/B)' = 1$ and therefore (taking into account the asymptotics) must be of the form

$$A(r) = 1 - \frac{2M}{r}; \quad B(r) = \left(1 - \frac{2M}{r}\right)^{-1}, \tag{9}$$

for some constant M.

4. **Tolman-Oppenheimer-Volkoff equations.** We consider the metric ansatz (6) from the previous exercise but this time we will impose Einstein's equations with a non-vanishing stress-energy tensor $T_\mu^\nu = \mathrm{diag}(\rho(r), -p(r), -p(r), -p(r))$.

(a) Write Einstein's equations (with $G = c = 1$) in the form $R_{\mu\nu} = -8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$ and derive the equations for R_{tt}, R_{rr} and $R_{\theta\theta}$.

(b) Using (8), show that

$$M'(r) = \frac{r^2}{4}\left(\frac{R_{tt}}{A} + \frac{R_{rr}}{B}\right) + \frac{1}{2}R_{\theta\theta} = 4\pi r^2\rho \tag{10}$$

where we defined M by $B = (1 - 2M/r)^{-1}$.

(c) Use the energy momentum conservation $D_\nu T_\mu^\nu = 0$ and (7) to derive

$$\frac{dp}{dr} = -(\rho + p)\frac{A'}{A} = -\frac{(\rho + p)(M + 4\pi p r^3)}{r^2(1 - 2M/r)}. \tag{11}$$