Exercise sheet 6: 21 Apr

NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 27 Apr, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **A charged particle falling into a charged black hole.** The action of a particle of mass m and charge q is

$$S[x,e] = \int d\lambda L(x,\dot{x},e) = \int d\lambda \left(\frac{1}{2} e^{-1} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} - \frac{1}{2} m^2 e - q\dot{x}^\mu A_\mu(x) \right), \tag{1}$$

where A_μ is the electromagnetic potential and $e(\lambda)$ is the einbein.

(a) Show that if we solve the equation of motion for e and plug it back into $S[x,e]$ that we get the more conventional action,

$$S[x] = -\int d\lambda\, m\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{2}$$

for $q = 0$. We will use (1) since it is more convenient computationally.

(b) Show that if ξ^μ is a Killing vector field and $\xi^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \xi^\nu = 0$, that $S[x,e]$ is invariant under $\delta x^\mu = \xi^\mu$.

(c) Show that the corresponding conserved quantity (the Noether charge) is $\xi^\mu(m\dot{x}_\mu - qA_\mu)$, where we chose to normalize $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$.

(d) Apply this to the Killing field $\partial/\partial t$ of the Reissner-Nordström metric to show that

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi r^2}\right)\dot{t} = E - \frac{qQ}{4\pi m r}, \tag{3}$$

where $E = -\xi^\mu(m\dot{x}_\mu - qA_\mu)/m$ is the "energy".

(e) Show that for a radially infalling particle,

$$\dot{r}^2 + A(r) - (E - \frac{qQ}{4\pi r m})^2 = 0. \tag{4}$$

(f) For which values of the ratio q/m will the particle hit the singularity? What happens for $E = 1$ and $q/m = Q/M = \sqrt{4\pi}$?

2. **Charged multi black hole solution.** The Majumdar-Papapetrou metric is given by

$$ds^2 = -\frac{1}{U(x)^2} dt^2 + U(x)^2 \delta_{ab} dx^a dx^b, \tag{5}$$

where δ_{ab} denotes the Euclidean metric on $\mathbb{R}^3$ and $U(x)$ is a time-independent function satisfying Laplace equation

$$\Delta U \equiv \delta^{ab} \frac{\partial^2 U}{\partial x^a \partial x^b} = 0. \tag{6}$$

We assume a static electromagnetic potential given by $A_\mu = \delta_{\mu 0} U^{-1}/\sqrt{4\pi}$.

(a) Show that A_μ solves the Maxwell equations $\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0$ in absence of sources (we derived this expression on exercise sheet 3).

A straightforward calculation shows that the metric satisfies the Einstein equations coupled to the electromagnetic field A_μ (do it if you don't believe).

(a) Show that a particular class of solutions to Laplace equation is given by

$$U(x) = 1 + \sum_{i=1}^{n} \frac{m_i}{\sqrt{(x - x_{(i)}) \cdot (x - x_{(i)})}}, \tag{7}$$

corresponding to the classical Newtonian potential of n point masses at $x = x_{(i)}$ and mass m_i. Note that the metric is well-defined except at the points $x = x_{(i)}$.

(b) Use Gauss law to compute the charges which the point masses must have. What do the points therefore represent?

(c) Show that for the case of a single point mass, the metric (5) is that of the extremal Reissner-Nordström black hole (after a suitable coordinate transformation).