Exercise sheet 9: 18 May

NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 25 May, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **Total energy emission by Hawking radiation.** In this exercise we will try to obtain a quantitative estimate for the amount of energy U per unit time emitted to spatial infinity by a Schwarzschild black hole of mass M. Recall that the black hole emits a pure black body spectrum of temperature $T = 1/(8\pi k M)$ (as measured by an observer at infinity).

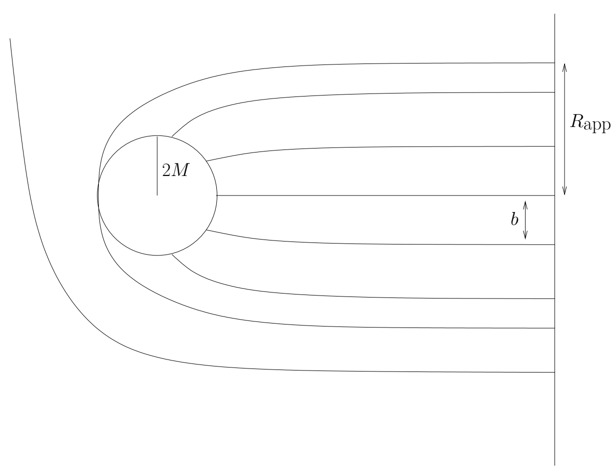
   Neglecting curvature and considering only massless bosonic radiation, the Stefan-Boltzmann law gives the energy emission U per unit of time for a black body of temperature T and area A,

   $$U = n_b \sigma A T^4, \tag{1}$$

   where $\sigma = \pi^2 k^4/120$ (we use $\hbar = c = 1$) and n_b is the number of independent bosonic degrees of freedom.

   (a) Calculate U in this case. What is n_b?

   If we want to incorporate the gravitational attraction, we should replace the area A by an apparent area: the area seen by an observer at spatial infinity. In general this area depends on the wave length λ of the emitted radiation. However if we assume λ to be small with respect to the radius $2M$ of the black hole, the photons will follow null geodesics and the apparent area A will be independent of λ.



   Let the energy $E = (1 - 2M/r)\dot{t}$ and angular momentum $L = r^2\dot{\phi}$ be the conserved quantities along a null geodesic. Analogous to previous exercises we find for null geodesics with $\theta = \pi/2$,

   $$\dot{r}^2 + \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) = E^2. \tag{2}$$

   (b) Show that the impact parameter b (see figure) is given by $|L/E|$.

(c) Show the apparent area $A = 4\pi R_{\text{app}}^2$ is given by

$$A = \frac{27}{4}(4\pi(2M)^2) \tag{3}$$

(d) Estimate the dominant wave length λ. How good is our assumption $\lambda \ll 2M$?

2. **Zeroth law of black hole dynamics.** Let's consider a static space-time metric $g_{\mu\nu}$. We can always write such a metric in the form

$$ds^2 = -g_{00}(x^i)dt^2 + g_{ij}(x^i)dx^i dx^j, \tag{4}$$

where $t = x^0$ and the indices i, j take the values $1, 2, 3$. In this case we can easily go to imaginary time $t = i\tau$ such that

$$ds^2 = g_{00}(x^i)d\tau^2 + g_{ij}(x^i)dx^i dx^j. \tag{5}$$

Show that if this metric is periodic in τ, i.e. that ds^2 is invariant under $\tau \to \tau + \beta(x^i)$, that the period $\beta(x^i)$ cannot depend on x^i. Hence, if we identify β at the horizon with the inverse surface gravity $2\pi/\kappa$, we conclude that κ is constant on the horizon.