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Phase Transitions for Continuous-Spin Ising Ferromagnets

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We study the comparison of continuous-spin ferromagnetic Ising models which differ only in their a priori single-spin weighting measures, and characterize the relationship of two even weighting measures ν', ν on $\mathbf{R}$ such that the spin expectations of any ferromagnet with single-spin weighting measure ν' are less than those of the same ferromagnet with single-spin measure ν. Combining these comparison results with an extension of Bortz and Griffiths' variant of the Peierls argument, we prove that any (nontrivial) continuous-spin ferromagnetic Ising model of dimension at least 2 with translation-invariant pair interaction is spontaneously magnetized at low temperature. Thus, phase transitions are generic in ferromagnetic Ising models of dimension at least 2.

1. INTRODUCTION

In this paper we investigate continuous-spin ferromagnetic Ising models. These continuous-spin models, which we rigorously define shortly, generalize the classical spin $\frac{1}{2}$ Ising models in that the spin variables σ_i are not restricted to the two values ± 1 but instead may assume any real value with some (temperature-independent) even a priori single-spin weighting measure ν. In Section 2 we study the relationship two single-spin measures ν', ν must have in

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order that for all Ising ferromagnets differing only in their single-spin measures ν', ν the Gibbs expectations obey

$$\left\langle \prod F_i(\sigma_i);\, \nu' \right\rangle \leqslant \left\langle \prod F_i(\sigma_i);\, \nu \right\rangle \tag{1.1}$$

for a natural class of functions $\prod F_i(\sigma_i)$ of the spins containing the monomials $\prod (\sigma_i)^{n_i}$. We reduce the comparison (1.1) to an abstract order relation $\prec$ among the probability measures on $[0, \infty)$, and characterize $\prec$ in terms of cumulative distribution functions. Section 3 combines a special case of the results in Section 2 with a generalization of an argument of Bortz and Griffiths [2] (which in turn extends the classic idea of Peierls [16]) to prove that all nontrivial nearest-neighbor continuous-spin Ising ferromagnets in two or more dimensions are long-range ordered at zero external field for sufficiently low temperature. We conclude from this that all (connected nontrivial) translation-invariant continuous-spin Ising ferromagnets with pair interactions are spontaneously magnetized at zero external field for sufficiently low temperature. Thus the appearance of a phase transition at low temperature in two or more dimensions is generic, while in contrast it is well known that finite-range models in one dimension are not spontaneously magnetized. Section 4 points out some further consequences of these general results on low-temperature phase transitions, including the existence of an equilibrium state with sharp phase interface in nearest-neighbor ferromagnets of three or more dimensions, and the spontaneous magnetization of anisotropic ferromagnetic plane rotors in two or more dimensions.

After the announcement of our work, Fröhlich *et al.* [6] developed a powerful new technique for establishing the existence of phase transitions. Though presently limited to three or more dimensions, their method reproduces our results on long-range order there.

We next define continuous-spin ferromagnetic Ising models, and summarize a few of their elementary properties. A continuous-spin Ising ferromagnet is a triple $(\mathscr{L}, H, \nu)$, where:

(1) The set of sites $\mathscr{L}$ is a denumerable set. We associate with each site $i \in \mathscr{L}$ a spin variable $\sigma_i \in \mathbf{R}$, and the product $\mathbf{R}^{\mathscr{L}} = \prod_{i \in \mathscr{L}} \mathbf{R}$ is called the configuration space.

(2) The Hamiltonian H is a formal polynomial in the spin variables σ_i, and the ferromagnetism assumption is that H has nonpositive coefficients. We write

$$H = - \sum_{K \in \mathscr{F}_0(\mathscr{L})} J_K \sigma_K, \qquad J_K \geqslant 0, \tag{1.2}$$

where the coefficients J_K are called couplings, $\mathscr{F}_0(\mathscr{L})$ is the set of finite families ("subsets" with repeated elements) in $\mathscr{L}$, and σ_K is by definition the product

$$\sigma_K = \prod_{i \in K} \sigma_i .$$

(We shall suppose the degree $\deg(H) = \sup\{| K |: J_K \neq 0\}$ is finite.)

(3) The single-spin measure ν is an even Borel probability measure on $\mathbf{R}$ which decays sufficiently rapidly that if d is the degree of H,

$$\int_{\mathbf{R}} \exp(a \mid \sigma \mid^d) \, d\nu(\sigma) < \infty \qquad \forall a \in \mathbf{R}. \tag{1.3}$$

As a simple illustration of this definition, we might take a two-site model

$$\mathscr{L} = \{1, 2\}$$

with Hamiltonian

$$H = -J_{\{1,1\}}(\sigma_1)^2 - J_{\{1,2\}}\sigma_1\sigma_2 - J_{\{1,2,2\}}\sigma_1(\sigma_2)^2$$

and single-spin measure

$$d\nu(\sigma) = \frac{2}{\Gamma(\frac{1}{4})} \exp(-\sigma^4) \, d\sigma.$$

If the set of sites $\mathscr{L}$ has finite cardinality $| \mathscr{L} | < \infty$ we conventionally replace $\mathscr{L}$ by Λ. Note that in a finite ferromagnet (Λ, H, ν), the Hamiltonian is an ordinary polynomial, and so is well defined as a function.

The linear term $-\sum_{i \in \mathscr{L}} J_i\sigma_i$ in (1.2) is commonly thought of as describing the effect of an external magnetic field, while higher-order terms are considered to arise from the mutual interactions of the spins. We usually recognize this distinction by writing $-\sum_{i \in \mathscr{L}} h_i\sigma_i$ in the Hamiltonian in place of $-\sum_{i \in \mathscr{L}} J_i\sigma_i$. A model is called connected if any pair of sites $i, j \in \mathscr{L}$ is connected by a finite chain $K_1, K_2, \ldots, K_n$ of families with $J_{K_1}, \ldots, J_{K_n} \neq 0$, $i \in K_1$, $j \in K_n$, and for all l, $K_l \cap K_{l+1} \neq \varnothing$. A pair interaction is a Hamiltonian of degree two. The models of principal physical and mathematical interest are those in which the set of sites $\mathscr{L}$ is $\mathbf{Z}^n$ and the Hamiltonian has properties connected with the geometrical nature of $\mathbf{Z}^n$. Typically, we shall require that the Hamiltonian be translation invariant:

$$J_K = J_{K+i} \qquad \forall K \in \mathscr{F}_0(\mathbf{Z}^n), \forall i \in \mathbf{Z}^n, \tag{1.4}$$

and finite range:

$$\operatorname{ran}(H) \equiv \sup_{\{K: J_K \neq 0\}} \operatorname{diam}(K) < \infty. \tag{1.5}$$

Here $K + i$ is the translate of the family K by $i \in \mathbf{Z}^n$, and the diameter $\operatorname{diam}(K)$ is $\sup_{i,j \in K} \| i - j \|$.

An example of a connected finite-range translation-invariant pair Hamiltonian is the nearest-neighbor interaction on $\mathbf{Z}^2$:

$$H = -J_1 \sum_{(i_1,i_2)\in\mathbf{Z}^2} \sigma_{(i_1,i_2)}\sigma_{(i_1+1,i_2)} - J_2 \sum_{(i_1,i_2)\in\mathbf{Z}^2} \sigma_{(i_1,i_2)}\sigma_{(i_1,i_2+1)} - h \sum_{i\in\mathbf{Z}^2} \sigma_i \,;$$
$$J_1, J_2 > 0,\ h \geqslant 0. \tag{1.6}$$

The Gibbs measure μ of a finite Ising ferromagnet (Λ, H, ν) taken at inverse temperature $\beta = 1/kT \in [0, \infty)$ is the probability measure on the configuration space $\mathbf{R}^\Lambda$ defined by

$$\mu(E) = Z^{-1} \int_E \exp[-\beta H(\sigma)] \prod_{i\in\Lambda} d\nu(\sigma_i), \qquad E \subset \mathbf{R}^\Lambda \text{ measurable}, \tag{1.7}$$

where Z is the partition function

$$Z = \int_{\mathbf{R}^\Lambda} \exp[-\beta H(\sigma)] \prod_{i\in\Lambda} d\nu(\sigma_i). \tag{1.8}$$

We indicate (thermal) expectations with respect to the Gibbs measure at inverse temperature β—in physical terms, averages over the canonical ensemble—by angular brackets $\langle\ ; H, \nu, \beta\rangle$, omitting the descriptive arguments H, ν, β when they are clear from context:

$$\langle f; H, \nu, \beta\rangle = \langle f\rangle = \int_{\mathbf{R}^\Lambda} f\, d\mu = Z^{-1} \int_{\mathbf{R}^\Lambda} f e^{-\beta H} \prod_\Lambda d\nu. \tag{1.9}$$

Physically, the sites Λ may be interpreted as the positions of atoms in a crystal, and the spin variable σ_i at each site $i \in \Lambda$ as a classical analog of the quantum-mechanical spin associated with the atom at i. The single-spin measure is a temperature-independent weight determined by internal properties of the atoms. A point σ in the configuration space $\mathbf{R}^\Lambda$ corresponds to a state of the system, and $H(\sigma)$ is the energy of that state. If we allow the crystal to exchange energy with a heat bath at reciprocal temperature β, then, in the limit as the bath becomes infinite, the equilibrium state of our crystal will be described by the canonical ensemble. Roughly speaking, this means that the probability of finding the system in some subset $E \subset \mathbf{R}^\Lambda$ of the configuration space is given by the Gibbs measure $\mu(E)$.

It follows from the ferromagnetism condition $J_K \geqslant 0$ and the symmetry of the single-spin measure that the moments of the Gibbs measure $\langle\sigma_A\rangle$, $A \in \mathscr{F}_\circ(\Lambda)$, obey the Griffiths inequalities [8, 19]

$$\text{(I):}\quad \langle\sigma_A\rangle \geqslant 0, \qquad A \in \mathscr{F}_\circ(\Lambda) \tag{1.10a}$$

$$\text{(II):}\quad \beta^{-1}\frac{\partial}{\partial J_B}\langle\sigma_A\rangle \equiv \langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle \geqslant 0, \qquad A, B \in \mathscr{F}_\circ(\Lambda). \tag{1.10b}$$

A more natural mathematical setting for the Griffiths inequalities is obtained by generalizing the definition given above of continuous-spin Ising ferromagnets. Specifically, we extend the definition of the Hamiltonian H from a formal polynomial to a formal sum with negative coefficients of finite products of functions $\prod H_i(\sigma_i)$, where each $H_i : \mathbf{R} \to \mathbf{R}$ has definite parity and is monotone increasing and nonnegative on $[0, \infty)$. (The parities of differing H_i need not be related.) To suppress technical integrability questions it is convenient to assume also that the H_i are bounded; thus, we need not impose decay restrictions on the single-spin measure. We call models with such Hamiltonians generalized continuous-spin Ising ferromagnets. In generalized ferromagnets the Griffiths inequalities (1.10) take on the form [15]

$$\text{(I):}\quad \left\langle \prod F_i(\sigma_i) \right\rangle \geqslant 0, \tag{1.11a}$$

$$\text{(II):}\quad \left\langle \prod F_i(\sigma_i) \times \prod G_j(\sigma_j) \right\rangle - \left\langle \prod F_i(\sigma_i) \right\rangle \left\langle \prod G_j(\sigma_j) \right\rangle \geqslant 0 \tag{1.11b}$$

for all families of (bounded) functions F_i, G_j having definite (unrelated) parity which are nonnegative and monotone increasing in $[0, \infty)$. We shall use this form of the Griffiths inequalities extensively in Section 2.

In an infinite model $(\mathscr{L}, H, \nu)$ we cannot use formula (1.7) to define the spin expectations $\langle \sigma_K \rangle$ directly because the Hamiltonian $H = -\sum J_K \sigma_K$ is only a formal polynomial in an infinite set of variables and makes no sense as a function. We use a limiting process to obviate this problem. If $\Lambda \subset \mathscr{L}$ is a finite subset, let the restriction H_Λ of H to Λ be

$$H_\Lambda = - \sum_{K \in \mathscr{F}_0(\Lambda)} J_K \sigma_K . \tag{1.12}$$

Thus, we set all spins outside Λ to zero: the zero boundary condition. If $A \in \mathscr{F}_0(\Lambda)$ we may use (1.7) to define the approximate moment

$$\langle \sigma_A \rangle_\Lambda = \langle \sigma_A \,; H_\Lambda , \nu, \beta \rangle. \tag{1.13}$$

By the second Griffiths inequality (1.10b), these approximate moments $\langle \sigma_A \rangle_\Lambda$ form an increasing net indexed by the finite subsets $\Lambda \subset \mathscr{L}$. For the finite-range translation-invariant pair interactions which are our primary interest, it is known that this net is bounded above [13, 18], and so converges to a finite limit

$$\langle \sigma_A \,; H, \nu, \beta \rangle = \lim_{\Lambda \to \infty} \langle \sigma_A \,; H_\Lambda , \nu, \beta \rangle, \tag{1.14}$$

the infinite-volume thermal expectation with the zero boundary condition. (For simplicity, we employ only this boundary condition.)

Consider a ferromagnet $(\mathbf{Z}^n, H, \nu)$ with finite-range translation-invariant pair interaction

$$H = -\sum_{i,j\in\mathbf{Z}^n} J_{(i-j)}\sigma_i\sigma_j - h\sum_{i\in\mathbf{Z}^n} \sigma_i . \tag{1.15}$$

By (1.10b), the spin expectations $\langle\sigma_A\rangle$ are monotone increasing in the external field h. When h vanishes the Hamiltonian H is invariant under simultaneous reversal of all spins $\sigma \to -\sigma$, so the odd moments $\langle\prod_{\alpha=1}^{2m+1} \sigma_{i_\alpha}\rangle$ are zero. The model $(\mathbf{Z}^n, H, \nu)$ is called spontaneously magnetized at zero external field if the magnetization $\langle\sigma_i\rangle$ is discontinuous in h at $h = 0$:

$$\lim_{h\downarrow 0^+}\langle\sigma_i\rangle \equiv m_s > 0. \tag{1.16}$$

While this behavior is clearly impossible in finite models, if the dimension $n \geqslant 2$ we shall show that for infinite models at low temperature it is inescapable. A closely related phenomenon is long-range order: $(\mathbf{Z}^n, H, \nu)$ is long-range ordered at zero external field if there exists $L > 0$ such that

$$\langle\sigma_i\sigma_j\rangle|_{h=0} \geqslant L \qquad \forall i, j \in \mathbf{Z}^n. \tag{1.17}$$

In Section 3 we derive spontaneous magnetization as a consequence of long-range order.

2. Inequalities

Working in the framework of the generalized continuous-spin Ising ferromagnets defined in connection with (1.11), we analyze the relationship two even (single-spin) measures must have in order that for all Hamiltonians H,

$$\left\langle\prod_{i\in\Lambda} F_i(\sigma_i);\, H, \nu'\right\rangle \leqslant \left\langle\prod_{i\in\Lambda} F_i(\sigma_i);\, H, \nu\right\rangle \tag{2.1}$$

for all (bounded) functions $F_i : \mathbf{R} \to \mathbf{R}$ of definite parity which are nonnegative on the right half-line $[0, \infty)$ and monotone increasing there. We show that (2.1) holds for all (generalized) ferromagnets (Λ, H, ν'), (Λ, H, ν) if and only if it holds for all (generalized) one-site ferromagnets ($|\Lambda| = 1$). We restate the condition that (2.1) hold for one-site models in terms of an abstract order relation $\prec$ on the set of probability measures on $[0, \infty)$, and complete our analysis by establishing that $\nu_1 \prec \nu_2$ if and only if $\nu_1[x, \infty)/\nu_2[x, \infty)$ is a monotone decreasing function of the lower limit x of the interval $[x, \infty)$. We conclude with some immediate applications of these results.

Throughout this section we deal with inequalities conveniently expressed in terms of ratios of nonnegative quantities: $a_1/b_1 \leqslant a_2/b_2$ where b_1 or b_2 may vanish. We always interpret such an inequality as the cross-multiplied version $a_1 b_2 \leqslant a_2 b_1$.

Consider a one-site ferromagnet whose negative Hamiltonian $(-H)\colon \mathbf{R} \to \mathbf{R}$ has odd parity and is, by definition, monotone increasing, nonnegative, and bounded on $[0, \infty)$, and let $F\colon \mathbf{R} \to \mathbf{R}$ also have these characteristics. In this situation inequality (2.1) becomes

$$\frac{\int_{[0,\infty)} f(\sigma)\, g(\sigma)\, d\hat{\nu}'(\sigma)}{\int_{[0,\infty)} g(\sigma)\, d\hat{\nu}'(\sigma)} \leqslant \frac{\int_{[0,\infty)} f(\sigma)\, g(\sigma)\, d\hat{\nu}(\sigma)}{\int_{[0,\infty)} g(\sigma)\, d\hat{\nu}(\sigma)} \tag{2.2}$$

where f and g are the restrictions of F and $\sinh(-H)$ to $[0, \infty)$, $\hat{\nu}$ is the measure on $[0, \infty)$ given by

$$\hat{\nu}(E) = 2\nu(E) - \nu(\{0\})\, \delta(E), \qquad E \subset [0, \infty), \tag{2.3}$$

and $\hat{\nu}'$ is defined similarly. (The measure δ is the point mass at zero.) Other choices of parities for F and H yield the same form as (2.2), though the relationship of $g(\sigma)$ to $H(\sigma)$ may differ. As F and H vary subject to the stated constraints, f and g range over the set of all monotone increasing nonnegative bounded functions on $[0, \infty)$. Thus we are lead to define the order relation $\prec$ among the probability measures on $[0, \infty)$ by

$$\nu_1 \prec \nu_2 \Leftrightarrow \frac{\int_{[0,\infty)} f \cdot g\, d\nu_1}{\int_{[0,\infty)} g\, d\nu_1} \leqslant \frac{\int_{[0,\infty)} f \cdot g\, d\nu_2}{\int_{[0,\infty)} g\, d\nu_2} \qquad \forall f, g \in \mathscr{M}, \tag{2.4}$$

where of course

$$\mathscr{M} = \{f\colon [0, \infty) \to [0, \infty) \mid \|f\|_\infty < \infty \;\&\; x \leqslant y \Rightarrow f(x) \leqslant f(y)\}. \tag{2.5}$$

Note that if $\nu_1, \nu_2 \prec \nu_3$ then any convex combination $\rho = \lambda\nu_1 + (1 - \lambda)\nu_2$ also obeys $\rho \prec \nu_3$.

Clearly, the condition that (2.1) hold for one-site generalized ferromagnets is just $\hat{\nu}' \prec \hat{\nu}$. We now show that $\hat{\nu}' \prec \hat{\nu}$ actually implies (2.1) for arbitrary generalized ferromagnets.

Proposition 2.1. *Let ν', ν be even (Borel) probability measures on* $\mathbf{R}$. *Then*

$$\left\langle \prod_{i\in\Lambda} F_i(\sigma_i); H, \nu' \right\rangle \leqslant \left\langle \prod_{i\in\Lambda} F_i(\sigma_i); H, \nu \right\rangle \tag{2.6}$$

for all (bounded) functions $F_i : \mathbf{R} \to \mathbf{R}$ of definite parity, nonnegative, and monotone increasing on $[0, \infty)$, and all finite generalized ferromagnets (Λ, H, ν'), (Λ, H, ν) if and only if

$$\hat{\nu}' \prec \hat{\nu}. \tag{2.7}$$

Remark. The boundedness assumptions we have made on the F_i and (implicitly) on H are for technical convenience only, and are easily removed by a limiting argument when ν', ν decay sufficiently rapidly to make the desired integrals finite.

Proof. From the preceding discussion, it is obvious that (2.6) implies (2.7). To establish the converse, we show that in an Ising model further generalized so that the single-spin measures are permitted to be different at different sites, the replacement of ν' by ν at a single site causes the expectations $\langle \prod F_i \rangle$ to increase. The proposition follows by successively applying this result to each site in the model.

Consider a ferromagnet on Λ with Hamiltonian H and single-spin measure ν_i at each site $i \in \Lambda$. Select a distinguished site $1 \in \Lambda$ at which we assume the single-spin measure is ν'. We want to show

$$(Z')^{-1} \int_{\mathbf{R}^\Lambda} \prod F_i(\sigma_i) \exp[-\beta H(\sigma)]\, d\nu'(\sigma_1) \prod_{i \neq 1} d\nu_i(\sigma_i)$$
$$\leqslant Z^{-1} \int_{\mathbf{R}^\Lambda} \prod F_i(\sigma_i) \exp[-\beta H(\sigma)]\, d\nu(\sigma_1) \prod_{i \neq 1} d\nu_i(\sigma_i). \tag{2.8}$$

Rewrite these expectations to isolate the dependence on ν', ν:

$$\left\langle \prod F_i\, ; \nu' \right\rangle = \int_{[0,\infty)} \left\langle \prod F_i \right\rangle_s Z(s)\, d\hat{\nu}'(s) \Big/ \int_{[0,\infty)} Z(s)\, d\hat{\nu}'(s); \tag{2.9a}$$

$$\left\langle \prod F_i\, ; \nu \right\rangle = \int_{[0,\infty)} \left\langle \prod F_i \right\rangle_s Z(s)\, d\hat{\nu}(s) \Big/ \int_{[0,\infty)} Z(s)\, d\hat{\nu}(s), \tag{2.9b}$$

where the functions $Z(s)$ and $\langle \prod F_i \rangle_s$ are defined by

$$Z(s) = \int_{\mathbf{R}^{\Lambda-1}} \tfrac{1}{2} \sum_{\sigma_1 = \pm s} \exp[-\beta H(\sigma)] \prod_{i \neq 1} d\nu_i(\sigma_i) \tag{2.10}$$

$$\left\langle \prod F_i \right\rangle_s = Z(s)^{-1} \int_{\mathbf{R}^{\Lambda-1}} \tfrac{1}{2} \sum_{\sigma_1 = \pm s} \left[\left(\prod_{i \in \Lambda} F_i(\sigma_i) \right) \exp[-\beta H(\sigma)] \right] \prod_{i \neq 1} d\nu_i(\sigma_i). \tag{2.11}$$

$Z(s)$ and $\langle \prod F_i \rangle_s$ have simple interpretations: they are the partition function and expectation of $\prod_{i \in \Lambda} F_i(\sigma_i)$ in the model where the single-spin measure at site 1 is the Bernoulli measure $\frac{1}{2}[\delta(\sigma_1 + s) + \delta(\sigma_1 - s)]$.

Rescale the variable σ_1 by a factor of s, so that the s-dependent measure $\frac{1}{2}[\delta(\sigma_1 + s) + \delta(\sigma_1 - s)]$ is replaced by the s-independent measure $\frac{1}{2}[\delta(\sigma_1 + 1) + \delta(\sigma_1 - 1)]$. A typical term $\prod H_i(\sigma_i)$ in the Hamiltonian becomes $H_1(s)\, \sigma_1^p \prod_{i \neq 1} H_i(\sigma_i)$, $p = 0$ or 1 being the parity of H_1 ; thus, a coupling $H_1(s)$ which increases in s effectively appears. The product $\prod_{i \in \Lambda} F_i(\sigma_i)$ transforms similarly. The Griffiths inequalities (1.11), which hold for the models we are considering, imply that partition functions and appropriate expectations in generalized ferromagnets are increasing functions of their couplings. Consequently, $Z(s)$ and $\langle \prod F_i \rangle_s$ lie in the set $\mathcal{M}$ of nonnegative increasing functions on $[0, \infty)$. Applying the definition (2.4) of $\hat{\nu}' \prec \hat{\nu}$ to representation (2.9), the conclusion (2.8) is immediate. Q.E.D.

We next turn to a characterization of the order relation $\prec$. For integration purposes, an increasing function $f \in \mathcal{M}$ is essentially a linear combination of characteristic functions (staircase):

$$f \sim \sum_{\alpha=1}^{m} f_\alpha \cdot \chi_{[y_\alpha, \infty)}, \qquad f_\alpha \in [0, \infty) \,\&\, 0 \leqslant y_1 \leqslant \cdots \leqslant y_m . \tag{2.12}$$

Taking $f = \chi_{[y,\infty)}$ and $g = \chi_{[x,\infty)}$ in (2.4) with $x \leqslant y$ ($x \geqslant y$ is trivial), we see that a necessary condition for $\nu_1 \prec \nu_2$ is

$$\frac{\nu_1[y, \infty)}{\nu_2[y, \infty)} \leqslant \frac{\nu_1[x, \infty)}{\nu_2[x, \infty)}, \qquad x \leqslant y; \tag{2.13}$$

that is, $\nu_1[x, \infty)/\nu_2[x, \infty)$ decreases. Even though (2.4) is not linear in g, this condition is also sufficient.

PROPOSITION 2.2. *Define the order relation $\prec$ among the probability measures on $[0, \infty)$ by (2.4). Then $\nu_1 \prec \nu_2$ if and only if $Q(x) \equiv \nu_1[x, \infty)/\nu_2[x, \infty)$ is a monotone decreasing function.*

Remark. In accord with our comments at the beginning of this section, by monotone decrease of Q we mean $x \leqslant y$ implies

$$\nu_1[x, \infty) \cdot \nu_2[y, \infty) \geqslant \nu_1[y, \infty) \cdot \nu_2[x, \infty).$$

Proof. Decrease of Q follows from $\nu_1 \prec \nu_2$ by the comments leading to (2.13). We now show conversely that decrease of Q also implies $\nu_1 \prec \nu_2$.

By linearity and approximation, it suffices to prove (2.4) when f is a characteristic function $f = \chi_{[y,\infty)}$. Decompose g by $\chi_{[y,\infty)}$:

$$g = \chi_{[0,y)} \cdot g + \chi_{[y,\infty)} \cdot g \equiv g^- + g^+. \tag{2.14}$$

After a cancellation, (2.4) becomes

$$\frac{\int g^+ \, d\nu_1}{\int g^+ \, d\nu_2} \leqslant \frac{\int g^- \, d\nu_1}{\int g^- \, d\nu_2}. \tag{2.15}$$

Again by linearity and approximation, it suffices to verify (2.15) when $g^\pm$ are characteristic functions:

$$\begin{aligned} g^- &= \chi_{[x^-, y)}, \\ g^+ &= \chi_{[x^+, \infty)}, \qquad x^- \leqslant y \leqslant x^+. \end{aligned} \tag{2.16}$$

Invoking (2.16), (2.15) becomes

$$\frac{\nu_1[x^+, \infty)}{\nu_2[x^+, \infty)} \leqslant \frac{\nu_1[x^-, \infty) - \nu_1[y, \infty)}{\nu_2[x^-, \infty) - \nu_2[y, \infty)}, \qquad x^- \leqslant y \leqslant x^+. \tag{2.17}$$

Since $\nu_1[x, \infty)/\nu_2[x, \infty)$ is decreasing we need only consider $x^+ = y$ in (2.17). But when $x^+ = y$, (2.17) reduces by cancellation to

$$\frac{\nu_1[y, \infty)}{\nu_2[y, \infty)} \leqslant \frac{\nu_1[x^-, \infty)}{\nu_2[x^-, \infty)}, \tag{2.18}$$

which is our hypothesis. Q.E.D.

Observe that by Proposition 2.2, once given ν_2 the initial segment $\{\nu_1 \mid \nu_1 \prec \nu_2\}$ is parameterized by the set of monotone decreasing (continuous from the left) functions Q: $[0, \infty) \to [0, \infty)$, $Q(0) = 1$. (Define the distribution function $\nu_1[x, \infty)$ to be $Q(x)\, \nu_2[x, \infty)$.) To further illustrate the relation $\prec$ we point out three examples of pairs ν_1, ν_2 with $\nu_1 \prec \nu_2$.

EXAMPLE 2.1. Let ν_2 be arbitrary, and let G: $[0, \infty) \to [0, \infty)$ be any nonnegative monotone decreasing function (normalized such that $\int G d\nu_2 = 1$). If we take

$$\nu_1 = G \cdot \nu_2 \tag{2.19}$$

then $\nu_1 \prec \nu_2$ because if $x \leqslant y$

$$\begin{aligned} \frac{\int_{[x,y)} G\, d\nu_2}{\int_{[x,y)} d\nu_2} \geqslant G(y) \geqslant \frac{\int_{[y,\infty)} G\, d\nu_2}{\int_{[y,\infty)} d\nu_2} &\Rightarrow \frac{\int_{[x,\infty)} G\, d\nu_2}{\int_{[x,\infty)} d\nu_2} \\ = \frac{\int_{[x,y)} G\, d\nu_2 + \int_{[y,\infty)} G\, d\nu_2}{\int_{[x,y)} d\nu_2 + \int_{[y,\infty)} d\nu_2} &\geqslant \frac{\int_{[y,\infty)} G\, d\nu_2}{\int_{[y,\infty)} d\nu_2} \end{aligned} \tag{2.20}$$

since the extra terms cancel; thus $\nu_1[x, \infty)/\nu_2[x, \infty)$ decreases.

EXAMPLE 2.2. Let ν_2 be arbitrary, let $c \in [0, \infty)$ be arbitrary, and let ρ be any (positive) measure supported in $[0, c]$ which is normalized such that $\rho[0, c] = \nu_2(c, \infty)$. If we take

$$\nu_1 = \chi_{[0,c]} \cdot \nu_2 + \rho, \tag{2.21}$$

where $\chi_{[0,c]}$ is the characteristic function of $[0, c]$, then $\nu_1 \prec \nu_2$ because

$$\frac{\nu_1[x, \infty)}{\nu_2[x, \infty)} = 1 - \frac{\rho[0, x)}{\nu_2[x, \infty)} \tag{2.22}$$

is manifestly decreasing.

In Examples 2.1 and 2.2, ν_1 was obtained from ν_2 by increasing the density of ν_2 at all x^+ larger than some $y \in [0, \infty)$ and decreasing it at all $x^- < y$. The following example shows that this behavior is not required in general.

EXAMPLE 2.3. Let

$$\begin{aligned} \nu_1(x) &= \tfrac{1}{20}[11\,\delta(x) + 2\,\delta(x-1) + 7\,\delta(x-2)], \\ \nu_2(x) &= \tfrac{1}{20}\,[2\,\delta(x) + 4\,\delta(x-1) + 6\,\delta(x-2) + 8\,\delta(x-3)]. \end{aligned} \tag{2.23}$$

Then $\nu_1 \prec \nu_2$ by explicit calculation.

Further examples may be produced by taking convex combinations.

We now combine Propositions 2.1 and 2.2 to obtain

THEOREM 2.3. *Let ν', ν be even probability measures on* $\mathbf{R}$. *Then*

$$\left\langle \prod_{i\in\Lambda} F_i(\sigma_i);\, H, \nu' \right\rangle \leqslant \left\langle \prod_{i\in\Lambda} F_i(\sigma_i);\, H, \nu \right\rangle \tag{2.24}$$

for all families of (bounded) functions F_i such that each $F_i : \mathbf{R} \to \mathbf{R}$ has definite parity and is nonnegative and monotone increasing on $[0, \infty)$, *and all generalized ferromagnetic Hamiltonians H, if and only if $\hat{\nu}'[x, \infty)/\hat{\nu}[x, \infty)$ is a monotone decreasing function of the lower limit x of the interval $[x, \infty)$, i.e., $\hat{\nu}' \prec \hat{\nu}$. (Here $\hat{\nu}'$, $\hat{\nu}$ are defined by* (2.3).)

The proof of this theorem goes through with minor modifications to obtain a similar result for ferromagnetic plane rotors (with circularly symmetric single-spin measure). As a useful corollary we have the special case

COROLLARY 2.4. *Let (Λ, H, ν'), (Λ, H, ν) be two finite ferromagnetic Ising models differing only in their single-spin measures. If $Q(x) \equiv \hat{\nu}'[x, \infty)/\hat{\nu}[x, \infty)$ is monotone decreasing, then the moments of the Gibbs measure decrease when ν is replaced by ν':*

$$\langle \sigma_A\,; H, \nu' \rangle \leqslant \langle \sigma_A\,; H, \nu \rangle \qquad \forall A \in \mathscr{F}_0(\Lambda). \tag{2.25}$$

The examples cited above in connection with Proposition 2.2 also serve to illustrate Theorem 2.3 and Corollary 2.4. Conversely, we mention a pair ν', ν where one might naively surmise $\langle \sigma_A\,; \nu' \rangle \leqslant \langle \sigma_A\,; \nu \rangle$, though in fact this inequality is false. Take

$$\begin{aligned} \nu'(\sigma) &= \tfrac{1}{4}[2\,\delta(\sigma) + \delta(|\,\sigma\,| - 2)], \\ \nu(\sigma) &= \tfrac{1}{4}[\delta(|\,\sigma\,| - 1) + \delta(|\,\sigma\,| - 2)]. \end{aligned} \tag{2.26}$$

At first sight one might expect the downshift of probability mass to zero in ν' relative to ν would cause the moments of the Gibbs measure to decrease.

However, the condition $\hat{\nu}' \prec \hat{\nu}$ is not satisfied, and if we consider a one-site model with external field Hamiltonian $H = -h\sigma$, $h > 0$, the putative inequality $\langle \sigma^m; \nu' \rangle \leqslant \langle \sigma^m; \nu \rangle$ reduces to

$$2^m \overset{?}{\leqslant} \frac{\cosh(h)[1 + \cosh(2h)]}{\cosh(2h)[\cosh(h) - 1]} \qquad (m \text{ even}), \tag{2.27}$$

which is false for large m.

We conclude this section by indicating several consequences of the results we have obtained. Theorem 2.3 enables us to compare critical temperatures of spontaneously magnetized ferromagnets with even Hamiltonians: if $(\mathscr{L}, H, \nu')$ is spontaneously magnetized with critical inverse temperature β_c', and $\hat{\nu} \succ \hat{\nu}'$, then $(\mathscr{L}, H, \nu)$ is also spontaneously magnetized, and $\beta_c \geqslant \beta_c'$, For example, suppose ν is absolutely continuous with respect to Lebesgue measure near zero, and has bounded Radon–Nikodym derivative $[d\nu/d\sigma]$ in some small interval $[-c, +c]$. Then we may use Example 2.2 to compare ν with the restriction ν' of Lebesgue measure to some possibly smaller interval (uniform distribution). Since many ferromagnets with uniform single-spin distribution are known to be spontaneously magnetized [9], this comparison provides a simple proof of phase transitions in ferromagnets whose single-spin measure is reasonably smooth near zero. For estimates on the critical temperature using this method, see [20].

As a further application, note that if single-spin measure ν has compact support, then setting $c = \max(\operatorname{supp} \nu)$ we find by Example 2.2 that the spin expectations $\langle \sigma_A ; \nu \rangle$ are bounded above by the spin expectations $\langle \sigma_A ; \nu_c \rangle$, where ν_c is the two-point measure $\nu_c(\sigma) = \frac{1}{2}[\delta(\sigma + c) + \delta(\sigma - c)]$. Thus, after rescaling the couplings in the Hamiltonian, we see that models with compact spins are bounded above by related classical spin $\frac{1}{2}$ models. If in addition the Hamiltonian is a pair interaction, the F.K.G. inequality [5] for continuous-spin Ising ferromagnets [10] implies that all correlations $\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle$ are bounded in terms of two-point correlations $\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle$ [12]. When the external field vanishes the two-point correlation reduces to $\langle \sigma_i \sigma_j \rangle$. The bound $\langle \sigma_i \sigma_j ; \nu \rangle \leqslant \langle \sigma_i \sigma_j ; \nu_c \rangle$ then gives a simple proof of the exponential decay of the correlations $\langle \sigma_A \sigma_B ; \nu \rangle - \langle \sigma_A ; \nu \rangle \langle \sigma_B ; \nu \rangle$ in the separation of A and B at high temperature β^{-1}, since such decay is easily established for classical spin $\frac{1}{2}$ Ising ferromagnets [4].

3. Phase Transitions

In this section we establish the existence of phase transitions in continuous-spin ferromagnets taken at low temperature. Our key technical result is Lemma 3.1, where we derive long-range order at low temperature in nearest-neighbor models on $\mathbf{Z}^n$, $n \geqslant 2$, for a restricted class of single-spin measures. As the proof of this lemma is a technically complex but conceptually straightforward elaboration of Bortz and Griffiths' variant [2] of the Peierls argument, we defer it to the Appendix. In Theorem 3.2 we dispense with the restrictions of Lemma 3.1 on the single-spin measure by invoking the comparison results of Section 2. Thus, any nearest-neighbor Ising ferromagnet $(\mathbf{Z}^n, H, \nu)$ in at least two dimensions whose single-spin measure ν is not the δ-function is long-range ordered at zero external field for sufficiently low temperature β^{-1}. Theorem 3.3 combines a simple geometric argument using the second Griffiths inequality (1.10b) with some abstract functional analysis to deduce from Theorem 3.2 the more general result that, at zero external field and sufficiently low temperature, any (nontrivial) connected translation-invariant Ising ferromagnet with pair Hamiltonian is spontaneously magnetized.

LEMMA 3.1. *Let $(\mathbf{Z}^n, H, \nu)$ be a nearest-neighbor continuous-spin Ising ferromagnet in dimension $n \geqslant 2$ with Hamiltonian*

$$H = -\sum_{k \in \mathbf{Z}^n} \sum_{\alpha=1}^{n} J_\alpha \sigma_k \sigma_{k+1_\alpha}, \qquad J_\alpha > 0,\ 1_\alpha \equiv (\underbrace{0,..., 0, 1}_{\alpha}, \overbrace{0,..., 0}^{n-\alpha}). \tag{3.1}$$

If $\exists c \in (0, \infty)$ such that $\operatorname{supp} \nu \subset [-c, c]$, *and if $\exists \eta > 0$ such that for all (measurable) $E \subset (-c/3, c/3)$ the measure $\nu(E + \frac{2}{3} \cdot c)$ of the translate $E + \frac{2}{3} \cdot c$ obeys*

$$\nu(E + \tfrac{2}{3}c) \geqslant \eta \nu(E), \tag{3.2}$$

and if $\nu[\frac{3}{4}c, c] \neq 0$, then for sufficiently low temperature β^{-1} the model is long-range ordered: $\exists L > 0$ such that $\forall i, j \in \mathbf{Z}^n$,

$$\langle \sigma_i \sigma_j\, ; H, \nu, \beta \rangle \geqslant L. \tag{3.3}$$

(The infinite-volume limit is taken with the zero boundary condition.)

Proof. The proof of Lemma 3.1 is given in the Appendix. Q.E.D.

Combining this lemma with Theorem 2.3, we obtain

THEOREM 3.2. *Let $(\mathbf{Z}^n, H, \nu)$ be a nearest-neighbor continuous-spin Ising ferromagnet in dimension $n \geqslant 2$ with Hamiltonian*

$$H = -\sum_{k \in \mathbf{Z}^n} \sum_{\alpha=1}^{n} J_\alpha \sigma_k \sigma_{k+1_\alpha}, \qquad J_\alpha > 0,\ 1_\alpha \equiv (\underbrace{0,..., 0, 1}_{\alpha}, \overbrace{0,..., 0}^{n-\alpha}) \tag{3.4}$$

whose (even) single-spin measure ν is not the δ-function: $\nu \neq \delta$. If the temperature β^{-1} is sufficiently small, the model is long-range ordered: $\exists L > 0$ such that

$$\langle \sigma_i \sigma_j \rangle \geqslant L \qquad \forall i, j \in \mathbf{Z}^n. \tag{3.5}$$

(The infinite-volume limit is taken with the zero boundary condition.)

Proof. If the single-spin measure ν is a Bernoulli measure $\frac{1}{2}[\delta(c+\sigma)+\delta(c-\sigma)]$, $c>0$, let ρ be the trivial measure $\rho=0$. If ν is not a Bernoulli measure, then since $\nu\neq\delta$, $\exists c>0$ such that $0<\nu[-c,c]<1$. If $\nu(-c/3,c/3)=0$, define the measure ρ to be

$$\rho=\frac{1}{2}\left(\frac{1}{\nu[-c,c]}-1\right)\chi_{[-c,c]}\nu+\frac{1}{2}(1-\nu[-c,c])\frac{\delta_{-c}+\delta_c}{2} \tag{3.6}$$

where $\chi_{[c,c]}$ is the characteristic function of the interval $[-c,c]$ and $\delta_{\pm c}$ is the δ-function at $\pm c$. If $\nu(-c/3,c/3)\neq 0$, define the measure ρ by

$$\rho(E)=\frac{1-\nu[-c,c]}{4\cdot\nu(-c/3,c/3)}\cdot\left[\nu\left(\left[E\cap\left(-c,-\frac{c}{3}\right)\right]+\frac{2c}{3}\right)\right.$$

$$\left.+\nu\left(\left[E\cap\left(\frac{c}{3},c\right)\right]-\frac{2c}{3}\right)\right] \tag{3.7}$$

$$+\frac{1}{2}(1-\nu[-c,c])\frac{\delta_{-c}(E)+\delta_c(E)}{2}. \tag{3.7}$$

Formula (3.7) shifts a small multiple of the measure in the central third of $[-c,c]$ to the left and right thirds, and adds δ-functions at $\pm c$.

Set $\nu'=\chi_{[-c,c]}\cdot\nu+\rho$. By Corollary 2.4 as applied through Example 2.2, replacing ν by ν' causes the moments $\langle\sigma_A\rangle$ to decrease. But ν' obeys the hypothesis of Lemma 3.1, so by its conclusion at low temperature $\langle\sigma_i\sigma_j\rangle\geq L\ \forall i,j\in\mathbf{Z}^n$.

Q.E.D.

The long-range order of Theorem 3.2 gives rise to spontaneous magnetization in

THEOREM 3.3. *Let $(\mathbf{Z}^n,H,\nu)$ be a continuous-spin Ising ferromagnet in dimension $n\geq 2$ with connected translation-invariant pair Hamiltonian*

$$H=-\sum_{k,l\in\mathbf{Z}^n}J_{(k-l)}\sigma_k\sigma_l-h\sum_{k\in\mathbf{Z}^n}\sigma_k,\qquad J_k,h\geq 0 \tag{3.8}$$

whose (even) single-spin measure ν is not the δ-function: $\nu\neq\delta$. If the temperature β^{-1} is sufficiently small, the model is spontaneously magnetized at $h=0$:

$$\lim_{k\downarrow 0^+}\langle\sigma_i;H,\nu,\beta\rangle\equiv m_s>0. \tag{3.9}$$

(The infinite-volume limit is taken with the zero boundary condition.)

Proof. By translation invariance, take $i=0$. By connectedness (and translation invariance) $\exists k,l\in\mathbf{Z}^n$ linearly independent such that $J_k, J_l\neq 0$. Let $\mathscr{L}\subset\mathbf{Z}^n$ be the subset $\mathscr{L}=\{ak+bl\mid a,b\in\mathbf{Z}\}$. Reducing to zero all couplings in H except for J_k, J_l makes $\mathscr{L}$ a sublattice disconnected from the rest of $\mathbf{Z}^n$

isomorphic to a nearest-neighbor model on $\mathbf{Z}^2$. Since by the second Griffiths inequality (1.10b) this reduction in couplings decreases the moments $\langle \sigma_A \rangle$, it suffices to prove Theorem 3.3 when $(\mathbf{Z}^n, H, \nu)$ is a two-dimensional nearest-neighbor model.

It is not difficult to show [20] that the two-point function clusters whenever the magnetization $\langle \sigma_i \rangle$ is differentiable in the external field:

$$\lim_{\|i-j\|\to\infty} (\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle) = 0 \qquad \text{if } \frac{d}{dh} \langle \sigma_i \rangle \text{ exists.} \tag{3.10}$$

By the second Griffiths inequality (1.10b) the magnetization is monotone increasing in h, and so is differentiable almost everywhere. Consequently we may find a decreasing sequence $h_m \downarrow 0$ such that $\langle \sigma_i \sigma_j ; h_m \rangle \to \langle \sigma_i ; h_m \rangle \langle \sigma_j ; h_m \rangle$ as $\| i - j \| \to \infty$ for all h_m. Suppose our model is long-range ordered, as Theorem 3.2 assures us two-dimensional nearest-neighbor ferromagnets will be at low temperature. Then by (1.10b) and the long-range order (3.5),

$$\langle \sigma_i \sigma_j ; h_m \rangle \geqslant L \qquad \forall i, j \in \mathbf{Z}^n. \tag{3.11}$$

Taking first $\| i - j \| \to \infty$ and then $h_m \downarrow 0$ yields $m_s \geqslant L^{1/2}$. Q.E.D.

This theorem also may be proven directly, by modifying the proofs of Lemma 3.1 and Theorem 3.2 to show that an arbitrarily small (volume-independent) external field on the boundary of a sequence of regions growing suitably to infinity gives rise to a magnetized state.

In some contexts, such as quantum field theory, it is natural to consider continuous-spin Ising models $(\mathbf{Z}^n, H, \nu)$ whose Hamiltonians have the form

$$H = \sum_{k,l \in \mathbf{Z}^n} J_{(k-l)} (\sigma_k - \sigma_l)^2 - h \sum_{k \in \mathbf{Z}^n} \sigma_k , \qquad J_k , h \geqslant 0. \tag{3.12}$$

According to the definition given in Section 1, such models are not ferromagnetic because the coefficients of the self-interaction terms $(\sigma_k)^2$ in (3.12) are positive. In constructing the Gibbs measure these self-interaction terms may be absorbed into the single-spin measure ν. However, they introduce a temperature dependence into the redefined single-spin measure which causes it to become more concentrated near zero as β increases. This temperature dependence opposes the tendency of the ferromagnetic cross terms $-2J_{(k-l)}\sigma_k\sigma_l$ in H to enhance the Gibbs probability of configurations far from the origin as β becomes larger, a tendency we have exploited in our proof of a phase transition. Consequently, one might anticipate no phase transition in an Ising model with Hamiltonian (3.12) if the initial single-spin measure were sufficiently peaked near zero, and this is indeed so. For example, Dunlop notes [3] that if $V: \mathbf{R} \to \mathbf{R}$ is an even C^1 function with $(1/x)\, V'(x) \geqslant \eta > 0\ \forall x \geqslant 0$, then the magnetization $\langle \sigma_i \rangle$ in a model with Hamiltonian (3.12) and (unnormalized) single-spin measure $d\nu(\sigma) = \exp[-V(\sigma)]\, d\sigma$ obeys

$$\langle \sigma_i\, ; H, \nu, \beta \rangle \leqslant h \cdot \frac{2}{\eta}. \tag{3.13}$$

Thus, no spontaneous magnetization is possible.

4. Further Developments

We point out two additional results following from the phase transition theorems of Section 3: the existence of an equilibrium state with a sharp phase interface for nearest-neighbor continuous-spin Ising ferromagnets in at least three dimensions, and the spontaneous magnetization of (connected translation-invariant) anisotropic continuous-spin plane rotors in at least two dimensions. Since we obtain these results by applying straightforward generalizations of existing methods [1, 11, 19] to the conclusions of the preceding section, we are content simply to state the theorems and sketch the techniques of proof. Greater detail is given in [20] and the references cited below.

Theorem 4.1. *Let* $(\mathbf{Z}^n, H, \nu)$ *be the nearest-neighbor ferromagnet in dimension* $n \geqslant 3$ *with Hamiltonian*

$$H = -\sum_{k \in \mathbf{Z}^n} \sum_{\alpha=1}^{n} J\sigma_k \sigma_{k+1_\alpha}, \qquad J > 0,\ 1_\alpha \equiv (\underbrace{0,\ldots, 0, 1}_{\alpha}, \overbrace{0,\ldots, 0}^{n-\alpha}), \tag{4.1}$$

and $\nu \neq \delta$. *Let* m_s *be the spontaneous magnetization of the nearest-neighbor ferromagnet* $(\mathbf{Z}^{n-1}, H', \nu)$ *in dimension* $(n-1)$ *with the same single-spin measure* ν, *coupling* J *and inverse temperature* β:

$$H' = -\sum_{k' \in \mathbf{Z}^{n-1}} \sum_{\alpha'=1}^{n-1} J\sigma'_{k'} \sigma'_{k'+1_{\alpha'}}. \tag{4.2}$$

Then for any inverse temperature β *there exists an equilibrium state* $\langle\ \rangle_{PI}$ *of* $(\mathbf{Z}^n, H, \nu)$ *such that*

$$\begin{aligned} \langle \sigma_i \rangle_{PI} &\geqslant m_s \quad \forall i = (i_1, \ldots, i_n) \in \mathbf{Z}^n \mid i_1 \geqslant 0, \\ \langle \sigma_i \rangle_{PI} &\leqslant -m_s \quad \forall i = (i_1, \ldots, i_n) \in \mathbf{Z}^n \mid i_1 < 0. \end{aligned} \tag{4.3}$$

Since for low enough temperature β^{-1} *the spontaneous magnetization* $m_s > 0$, *the state* $\langle\ \rangle_{PI}$ *then has a sharp phase interface along the hyperplane* $i_1 = 0$.

Proof (Sketch). The proof, which generalizes an argument of van Beijeren [1], is outlined in [19] and given in detail in [20]. The basic idea is to apply a positive external field h to all sites with first coordinate $i_1 \geqslant 0$, and the opposite field $(-h)$ to the sites with first coordinate $i_1 < 0$. Using the method of duplicate variables [7, 17, 19], one may show that the magnetization $\langle \sigma_i \rangle_{i_1 \geqslant 0}$ in the half of the model with positive external field h is larger than the magnetization $\langle \sigma'; H' - h \sum \sigma'_{k'} \rangle$ in the $(n-1)$-dimensional slice with Hamiltonian (4.2) and positive external field h. Similarly, the magnetization $\langle \sigma_i \rangle_{i_1 < 0}$ in the other half of the model is less than $-\langle \sigma'; H' - h \sum \sigma'_{k'} \rangle$. The theorem follows upon sending h to zero. Q.E.D.

Our final theorem deals with plane rotors, which we have not discussed in detail. A plane rotor differs from the Ising models we have analyzed in that the spin variable σ_i at site $i \in \mathscr{L}$ is not a real number but lies in the plane $\mathbf{R}^2$ with some circularly symmetric a priori single-spin weighting measure $d\nu = d\rho(r)\, d\theta$. The Hamiltonian remains a formal polynomial in the components σ_i^x, σ_i^y of the spins.

THEOREM 4.2. *Let $(\mathbf{Z}^n, H, d\rho(r)\, d\theta)$ be a continuous-spin ferromagnetic plane rotor in dimension $n \geqslant 2$ whose Hamiltonian H is an anisotropic connected translation-invariant pair interaction of the form*

$$H = -\sum_{k,l \in \mathbf{Z}^n} [J^x_{(k-l)} \sigma_k^x \sigma_l^x + \gamma_{(k-l)} J^x_{(k-l)} \sigma_k^y \sigma_l^y], \qquad J_k^x \geqslant 0,\ 0 \leqslant \gamma_k < 1. \quad (4.4)$$

If the radial measure ρ is not the δ-function δ, then for sufficiently low temperature β^{-1} the model is spontaneously magnetized in the x-direction:

$$m_s^x \equiv \lim_{h \downarrow 0^+} \left\langle \sigma_i^x; H - h \sum_{k \in \mathbf{Z}^n} \sigma_k^x, d\rho(r)\, d\theta, \beta \right\rangle > 0. \quad (4.5)$$

Proof (Sketch). An argument of Kunz *et al.* [11] employing the Ginibre inequality for plane rotors [7] shows that the spontaneous magnetization m_s^x in the x-direction of the model $(\mathbf{Z}^n, H, d\rho(r)\, d\theta)$ is bounded below by the spontaneous magnetization m_{Ising} of the Ising ferromagnet $(\mathbf{Z}^n, H_{\text{Ising}}, \nu)$, where

$$H_{\text{Ising}} = -\sum_{k,l \in \mathbf{Z}^n} J^x_{(k-l)} [1 - \gamma_{(k-l)}]\, \sigma_k \sigma_l \quad (4.6)$$

and ν is defined by

$$\nu(E) = \int_{E \times \mathbf{R}} d\rho(r)\, d\theta. \quad (4.7)$$

Since ρ is not the δ-function δ, neither is ν, so by Theorem 3.3,$0 < m_{\text{Ising}} \leqslant m_s^x$ for sufficiently low temperature. Q.E.D.

The restriction to anisotropic Hamiltonians (4.4) is not artificial; Theorem 4.2 is known to fail in two dimensions [14] for models with isotropic Hamiltonians

($\gamma_k = 1$). However, the recent work of Fröhlich *et al.* [6] shows that by advancing to three or more dimensions a spontaneous magnetization is obtained even for isotropic models.

Appendix

This Appendix is devoted to a proof of Lemma 3.1. Arguing with the second Griffiths inequality (1.10b) as in Theorem 3.3, we may reduce to the case when the dimension n is 2. Thus we must prove

LEMMA. *Let* $(\mathbf{Z}^2, H, \nu)$ *be a nearest-neighbor* Ising *ferromagnet with Hamiltonian*

$$H = -J \sum_{(l,m)\in\mathbf{Z}^2} [\sigma_{(l,m)}\sigma_{(l+1,m)} + \sigma_{(l,m)}\sigma_{(l,m+1)}], \qquad J > 0. \tag{A.1}$$

If $\exists c \in (0, \infty)$ *such that* $\operatorname{supp} \nu \subset [-c, c]$, *and if* $\exists \eta > 0$ *such that for all measurable* $E \subset (-c/3, c/3)$

$$\nu(E + \tfrac{2}{3}c) \geqslant \eta\nu(E) \tag{A.2}$$

and if $\nu[\tfrac{2}{3}c, c] \neq 0$, *then for sufficiently low temperature* β^{-1} $\exists L > 0$ *such that* $\forall i, j \in \mathbf{Z}^2$,

$$\langle \sigma_i\sigma_j\,;\, H, \nu, \beta\rangle \geqslant L. \tag{A.3}$$

(*The infinite-volume limit is taken with the zero boundary condition.*)

Proof. As much of this proof follows standard reasoning, we shall give the details in a condensed manner. We may assume without loss of generality that $c = 1$ and $\eta < 1$. Let χ_n be the characteristic function of the interval

$$\begin{aligned} n &= 1: \quad [\tfrac{1}{3}, 1],\\ n &= 2: \quad (-\tfrac{1}{3}, \tfrac{1}{3}),\\ n &= 3: \quad [-1, -\tfrac{1}{3}]. \end{aligned} \tag{A.4}$$

Estimate $\langle\sigma_i\sigma_j\rangle$ using these characteristic functions:

$$\begin{aligned} \langle\sigma_i\sigma_j\rangle &= \sum_{n,m} \langle\sigma_i\sigma_j\chi_m(\sigma_i)\,\chi_n(\sigma_j)\rangle = \sum_n \langle\sigma_i\sigma_j\chi_n(\sigma_i)\,\chi_n(\sigma_j)\rangle\\ &\quad + \sum_{m\neq n} \langle\sigma_i\sigma_j\chi_m(\sigma_i)\,\chi_n(\sigma_j)\rangle\\ &\geqslant \frac{1}{9}\,[\langle\chi_1(\sigma_i)\,\chi_1(\sigma_j)\rangle + \langle\chi_3(\sigma_i)\,\chi_3(\sigma_j)\rangle] - \frac{1}{9}\,\langle\chi_2(\sigma_i)\,\chi_2(\sigma_j)\rangle\\ &\quad - \sum_{m\neq n} \langle\chi_m(\sigma_i)\,\chi_n(\sigma_j)\rangle. \end{aligned} \tag{A.5}$$

We show that by choosing β sufficiently large (independently of i, j) the two negative terms in (A.5) may be made as close to zero as desired, so that the first must approach $\frac{2}{9}$ and thus give the desired lower bound.

The term $\langle \chi_2(\sigma_i)\, \chi_2(\sigma_j) \rangle \leqslant \langle \chi_2(\sigma_i) \rangle$ is easily disposed of. By inequality (1.11b), $\langle \chi_2(\sigma_i) \rangle$ increases when any coupling between two sites is decreased. Thus if we consider a model with just two sites i, i' at inverse temperature β with coupling J and single-spin measure ν,

$$\langle \chi_2(\sigma_i) \rangle \leqslant \langle \chi_2(\sigma_i) \rangle', \tag{A.6}$$

where the prime on the right indicates the expectation in the two-site system. By a simple estimate, the condition $\nu[\frac{3}{4}, 1] > 0$ in the hypothesis assures $\lim_{\beta \to \infty} \langle \chi_2(\sigma_i) \rangle' = 0$, so that $\langle \chi_2(\sigma_i)\, \chi_2(\sigma_j) \rangle \leqslant \langle \chi_2(\sigma_i) \rangle'$ approaches zero as β is increased in the original model.

The term $\sum_{m \neq n} \langle \chi_m(\sigma_i)\, \chi_n(\sigma_j) \rangle$ is controlled by an extension of the ideas of Bortz and Griffiths [2], who considered in a somewhat different context the case when ν was Lebesgue measure restricted to $[-1, 1]$. By the spin-reversal symmetry of the Gibbs measure it suffices to show that $\langle \chi_1(\sigma_i)[\chi_2(\sigma_j) + \chi_3(\sigma_j)] \rangle$ becomes small for large β. To accomplish this, we shall prove that if $\Lambda \ni i, j$ is sufficiently large, then $\forall \epsilon > 0\ \exists \beta_\epsilon$ independent of i, j, Λ such that

$$\beta > \beta_\epsilon \Rightarrow \langle \chi_1(\sigma_i)[\chi_2(\sigma_j) + \chi_3(\sigma_j)] \rangle < \epsilon. \tag{A.7}$$

Regard $\mathbf{Z}^2$ as a subset of $\mathbf{R}^2$, and associate with each $i \in \mathbf{Z}^2$ a closed unit square $\Delta_i \subset \mathbf{R}^2$ centered at i. If $\Lambda \subset \mathbf{Z}^2$, define $\mathbf{\Lambda} \subset \mathbf{R}^2$ by $\mathbf{\Lambda} = \bigcup_{i \in \Lambda} \Delta_i$. Given a configuration $\sigma \in [-1, 1]^\Lambda$, we call the spin at site $k \in \Lambda$ plus (+) if $\sigma_k \in [\frac{1}{3}, 1]$ and minus (—) if $\sigma_k \in [-1, \frac{1}{3})$. Break up $\mathbf{\Lambda}$ into + and — connected components by saying that two squares Δ_k, Δ_l are in the same + (—) connected component if their spins are both + (—) and they are connected by a chain of nearest-neighbor squares with all + (—) spins. A border B associated with the configuration σ is defined as (the closure in $\mathbf{\Lambda}$ of) a connected component of the boundary taken in the interior of $\mathbf{\Lambda}$ of a $\pm$ connected component. Note that a border must either be a closed polygon or have both ends on $\partial\mathbf{\Lambda}$. Thus B separates $\mathbf{\Lambda}$ into two connected components. A site k is called a circumference site if its unit square Δ_k has a side in B. If b is the length of B there are at most b circumference sites in each component. The circumference sites in one of the connected components must be either all + or all —, and in the other all — or all +. We call the + (—) component the one in which all sites are + (—). An example is shown in Fig. A.1.

Let $\Lambda \subset \mathbf{Z}^2$ be a square containing i, j which is so large that the inequality

$$\operatorname{dist}(\{i, j\}, \partial\mathbf{\Lambda}) \geqslant | i_1 - j_1 | + | i_2 - j_2 | \tag{A.8}$$

is satisfied by the corresponding square $\mathbf{\Lambda} \subset \mathbf{R}^2$. We shall show that if B is a border in $\mathbf{\Lambda}$ and $\mathscr{B} \subset [-1, 1]^\Lambda$ is the set of all configurations σ which have B as

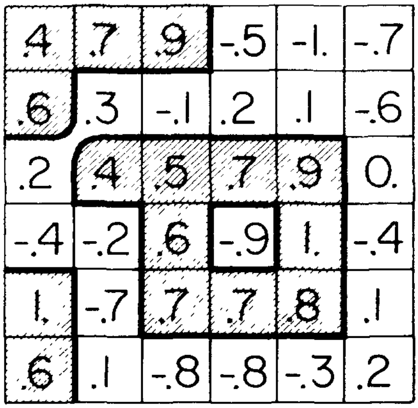

FIG. A.1. This example shows three + components (shaded area) and two — components. There are four borders, drawn with heavy black lines.

a border, then the Gibbs measure $P_B = Z^{-1} \int_{\mathscr{B}} e^{-\beta H} \prod_{\Lambda} d\nu$ of $\mathscr{B}$ decays exponentially in the length b of the border B:

$$P_B \leqslant 4(2/\eta)^b \, e^{-\beta J b/9}. \tag{A.9}$$

Taking estimate (A.9) as an assumption, let us see why long-range order follows.

If σ_i is $+$ and σ_j is $-$ in some configuration, then one of the borders B bounding the $+$ connected component containing i must separate i from j in Λ. Thus, if $B(i, j)$ is the set of all borders separating i from j in Λ, we have the inequality

$$\langle \chi_1(\sigma_i)[\chi_2(\sigma_j) + \chi_3(\sigma_j)] \rangle \leqslant \sum_{B \in B(i,j)} P_B \,. \tag{A.10}$$

A border may separate i from j in one of three ways: it may be a closed polygon with i in its interior and j in its exterior, it may be a closed polygon with j in its interior and i in its exterior, or it may have both endpoints on $\partial\Lambda$ and pass between i and j. The number of borders of length b enclosing either i or j is at most $b3^b$. Also, since the number of borders of length b containing a particular side of a particular square Δ_k is bounded by $b3^b$, and since any border separating i from j must pass through one of the $|i_1 - j_1| + |i_2 - j_2|$ intervening sides pointed out in Fig. A.2, the number of borders of length b separating i from j with endpoints on $\partial\Lambda$ is at most $(|i_1 - j_1| + |i_2 - j_2|)\, b3^b$. However, if the border B is long enough to separate i, j and to extend to $\partial\Lambda$, then by (A.8) we must have

$$b \geqslant \operatorname{dist}(\{i, j\}, \partial\Lambda) \geqslant |i_1 , - j_1| + |i_2 , - j_2|. \tag{A.11}$$

Combining these estimates, we find that the number $\#(b)$ of borders of length b separating i from j is at most

$$\#(b) \leqslant 2b^2 3^b. \tag{A.12}$$

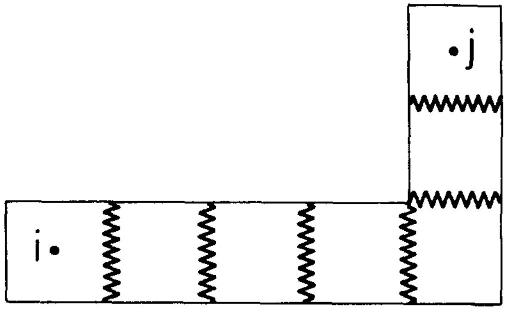

FIG. A.2. Any border separating i, j must pass through one of the jagged sides.

It now follows from (A.9), (A.10), and (A.12) that

$$\langle \chi_1(\sigma_i)[\chi_2(\sigma_j) + \chi_3(\sigma_j)] \rangle \leqslant \sum_{b=4}^{\infty} 8b^2 \left(\frac{6}{\eta}\right)^b e^{-\beta J b/9}. \tag{A.13}$$

Since the right-hand side of (A.13) becomes arbitrarily small for large β independent of i, j, and Λ, we will have long-range order once the exponential decay of P_B in b is established. We turn now to this problem.

We shall say that the spin σ_k at site k is in class n, $n = 1, 2, 3$, if

$$\begin{aligned} n = 1&: \quad \sigma_k \in [\tfrac{1}{3}, 1], \\ n = 2&: \quad \sigma_k \in (-\tfrac{1}{3}, \tfrac{1}{3}), \\ n = 3&: \quad \sigma_k \in [-1, -\tfrac{1}{3}]. \end{aligned} \tag{A.14}$$

Fix a particular border B. It separates Λ into two components Λ', Λ''. Let $\mathscr{B}' \subset [-1, 1]^\Lambda$ be the set of all configurations in which B appears as a border and Λ' is the $-$ component. Let C_B be the set of circumference sites of B in Λ', let $n \in \prod_{k \in C_B} \{2, 3\}$ be a multiindex, and let $\mathscr{B}_n \subset \mathscr{B}'$ be the set of configurations for which the spin at site $k \in C_B$ is in class n_k. Define the transformation $\tau_1 : \mathscr{B}_n \to [-1, 1]^\Lambda$ by

$$\begin{aligned} \tau_1(\sigma)_k &= \sigma_k \qquad \text{if } k \text{ is in the } + \text{ component } \Lambda'', \\ &= -\sigma_k + \tfrac{2}{3} \qquad \text{if } k \in C_B \;\&\; n_k = 2, \\ &= -\sigma_k \qquad \text{otherwise.} \end{aligned} \tag{A.15}$$

Define the transformation $\tau_2 : \mathscr{B}_n \to [-1, 1]^\Lambda$ by

$$\begin{aligned} \tau_2(\sigma)_k &= \sigma_k + \tfrac{2}{3} \qquad \text{if} \quad k \in C_B \;\&\; n_k = 2, \\ &= \sigma_k \qquad \text{otherwise.} \end{aligned} \tag{A.16}$$

Note that both τ_1 and τ_2 factor:

$$\tau_\alpha(\sigma)_k = \tau_{\alpha k}(\sigma_k), \qquad \alpha = 1, 2, \tag{A.17}$$

where $\tau_{\alpha k} : [-1, 1] \to [-1, 1]$ is determined from definitions (A.15), (A.16). Also, they are both one-to-one, so we may define the measures $\tau_\alpha{}^*\nu$ on $\mathscr{B}_n$ by

$$(\tau_\alpha{}^*\nu)(E) = \nu(\tau_\alpha E), \qquad E \subset \mathscr{B}_n\,, \alpha = 1, 2. \tag{A.18}$$

We claim that

$$\tau_\alpha{}^*\nu(E) \geqslant \eta^b \nu(E), \qquad E \subset \mathscr{B}_n\,, \alpha = 1, 2. \tag{A.19}$$

It suffices to verify this for rectangles $E = \prod_{k \in \Lambda} E_k\,, E_k \subset [-1, 1]$. If $k \in \mathbf{\Lambda}''$, $\tau_{\alpha k} E_k = E_k$ so $\nu(\tau_{\alpha k} E_k) = \nu(E_k)$. If $k \in \mathbf{\Lambda}'$ but is not a circumference site of class 2, $\tau_{\alpha k} E_k = \pm E_k$; by the evenness of ν, $\nu(\tau_{\alpha k} E_k) = \nu(E_k)$. If k is a circumference site of class 2 in $\mathbf{\Lambda}'$, $\tau_{\alpha k} E_k = \frac{2}{3} \pm E_k$; by the hypothesis of the lemma $\nu(\frac{2}{3} \pm E_k) \geqslant \eta \nu(\pm E_k) = \eta \nu(E_k)$. Since there are at most b circumference sites in Λ', inequality (A.19) must hold as claimed.

We finish the argument by following Bortz and Griffiths [2] in estimating $\int_{\mathscr{B}'} e^{-\beta H}\, d\nu$. They show that either

$$H_\Lambda(\tau_1 \sigma) \leqslant H_\Lambda(\sigma) - Jb/9 \tag{A.20a}$$

or

$$H_\Lambda(\tau_2 \sigma) \leqslant H_\Lambda(\sigma) - Jb/9. \tag{A.20b}$$

Let $\mathscr{B}_n{}^1 \subset \mathscr{B}_n$ be the set of all configurations in $\mathscr{B}_n$ such that (A.20a) holds, and let $\mathscr{B}_n{}^2 = \mathscr{B}_n - \mathscr{B}_n{}^1$. Then

$$\int_{\mathscr{B}'} e^{-\beta H(\sigma)}\, d\nu(\sigma) = \sum_{\alpha=1,2} \sum_{n \in \Pi_{C_B}\{2,3\}} \int_{\mathscr{B}_n{}^\alpha} e^{-\beta H(\sigma)}\, d\nu(\sigma). \tag{A.21}$$

But

$$\int_{\mathscr{B}_n{}^\alpha} e^{-\beta H(\sigma)}\, d\nu(\sigma) \leqslant \eta^{-b} \cdot e^{(-1/9)\beta J b} \cdot Z \tag{A.22}$$

since

$$Z \geqslant \int_{\tau_\alpha \mathscr{B}_n{}^\alpha} e^{-\beta H(s)}\, d\nu(s) = \int_{\mathscr{B}_n{}^\alpha} e^{-\beta H(\tau_\alpha \sigma)} d(\tau_\alpha{}^*\nu)(\sigma) \geqslant \eta^b e^{\beta J b/9} \int_{\mathscr{B}_n{}^\alpha} e^{-\beta H(\sigma)}\, d\nu(\sigma). \tag{A.23}$$

Using estimate (A.22) in (A.21) and summing over α and $\boldsymbol{n}$ we find

$$Z^{-1} \int_{\mathscr{B}'} e^{-\beta H(\sigma)}\, d\nu(\sigma) \leqslant 2 \cdot \left(\frac{2}{\eta}\right)^b e^{-\beta J b/9}. \tag{A.24}$$

If we take into account the fact that when the border B appears either of the components $\mathbf{\Lambda}'$, $\mathbf{\Lambda}''$ may be the $-$ component then we obtain estimate (A.9) for P_B . Q.E.D.

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