ABNORMAL DIFFUSION IN EHRENFEST'S WIND-TREE MODEL

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It is shown that in Ehrenfest's wind-tree model with overlapping scatterers, the mean square displacement for small densities ρ and long times t behaves like $4D_B(t/t_0)^{1-4\rho/3}$ where D_B is the Boltzmann diffusion coefficient.

In this letter we present a simple derivation of the abnormal asymptotic behavior of the mean square displacement $\Delta(t)$ of the moving particles in Ehrenfest's wind-tree model [1, 2] with randomly distributed (i.e., overlapping allowed) square scatterers ("trees"). Our result which is valid for long times t and small dimensionless densities $\rho = na^2$ (n is the number density of the scatterers and $2a$ the length of their diagonals), is consistent with recent computer experiments [3].

The assumption basic to our argument is the following: For sufficiently low densities, aside from the uncorrelated collisions taken into account by Boltzmann's Stosszahlansatz, it is only necessary to consider retracing events [2] due to reflections by two trees (see fig. 1). Rather than study $\Delta(t)$ directly, we shall discuss its derivative

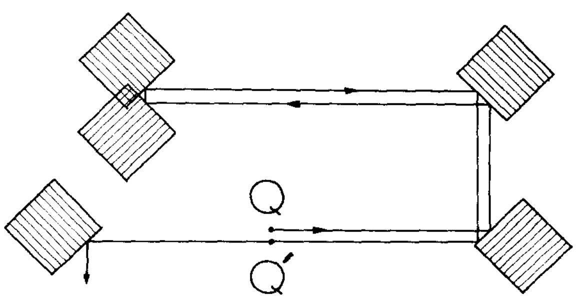

Fig. 1. Collision event with reflection.

$$\frac{1}{4}\frac{\mathrm{d}\Delta}{\mathrm{d}t} \equiv D(t) = \frac{1}{2}\int_0^t \mathrm{d}t' \langle \boldsymbol{v}(0)\cdot\boldsymbol{v}(t')\rangle. \tag{1}$$

Allowing for the moment uncorrelated collisions only and noting that from the first collision on, the average velocity is zero in this case, one finds immediately in the Boltzmann approximation

$$D_B(t) = \frac{1}{2}\int_0^t \mathrm{d}t'\, v^2 \exp(-t'/t_0) = D_B(1-\exp(-t/t_0)), \tag{2}$$

where $D_B = av/4\rho$ is the Boltzmann diffusion coefficient and $t_0 = a/2v\rho$ is the corresponding mean free time between collisions.

Consider next the modifications introduced by reflections. Let $P_0(t)$ be the fraction of particles which do not return to their starting point along a double path within the time t. (With reference to fig. 1, by return to Q we shall mean arrival at Q'.) The contributions of these particles to $D(t)$ will approach $D_B P_0(t)$ for $t \gg t_0$. If a particle does return in the above manner within t, however, its original contribution to eq. (1) is cancelled by a corresponding contribution from the returning path. In addition, after having passed Q' it builds up a *negative* contribution to $D(t)$ of (on the average) the same size as the original one. Defining $P_m(t)$ as the fraction of particles that return along a double path precisely m times within t, one can repeat the argument to find that $D(t) \approx D_B \Sigma_m (-1)^m P_m(t)$.

Noting that the decrease of $P_0(t)$ during dt must be caused by a reflection during the interval $[\frac{1}{2}t, \frac{1}{2}(t + dt)]$, one can write

$$dP_0(t) = -P_0(t) \int_0^a db\, n^2 a^2 v dt \exp(-3nbvt/2). \quad (3)$$

Here $n^2a^2 v db dt$ is the probability, to lowest nonvanishing order in n, that a reflection by two trees creates a double path with a width (absolute value) between b and $b + db$ during the time interval $\frac{1}{2}dt$. The damping $\exp(-\frac{3}{2}nbvt)$ represents the probability (the exponent again to lowest order in n) that the double path is not split up before the particle has returned to Q'. The asymptotic solution of eq. (3) for $t \gg t_0$ is $P_0(t) \approx (t/t_0)^{-2\rho/3}$.

A similar argument yields the following equation for $P_1(t)$

$$P_1(t) = \int_0^t dt' [-dP_0(t')/dt'] P_0(t-t'). \quad (4)$$

From eqs. (3) and (4) one readily deduces that $P_0(t) - P_1(t) \approx [1 + O(\rho)] (t/t_0)^{-\alpha(\rho)}$ with $\alpha(\rho) = \frac{4}{3}\rho + O(\rho^2)$. Furthermore, one can show that the sum $\Sigma_{m\geqslant 2}(-1)^m P_m(t)$ has the asymptotic form $f(\rho)(t/t_0)^{-\alpha(\rho)}$ where $f(\rho) \to 0$ when $\rho \to 0$. Thus, by integrating $P_0 - P_1$ with respect to time, one obtains the desired result: To lowest order in ρ and for $t \gg t_0$, the mean square displacement is given by

$$\Delta(t) \approx 4D_B (t/t_0)^{1-4\rho/3}. \quad (5)$$

1. The result (5) is consistent with the "experimental" data [3]. It is also in complete agreement with the systematic expansion to $O(\rho^2)$ of (essentially) $D^{-1}(t)$ found in ref. [2], if one, in accordance with the aim of the present letter, disregards $O(\rho^2)$ terms which are finite in the limit $t \to \infty$.

2. Any finite fraction of a could be used as an upper limit of integration in eq. (3) without affecting result (5). On the other hand, with *non*-overlapping trees n^2a^2 in eq. (3) is replaced by n^2ab for small b, and the result is a *normal* diffusion process, $\Delta(t) \sim 4Dt$. Thus the qualitative difference between the two cases is caused by very narrow, very long paths [2].

3. The reasoning of this letter can be refined [4] to yield the $O(\rho^2)$ term in the exponent of (5).

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References

[1] P. Ehrenfest, Collected scientific papers (North-Holland, 1959) p. 229.
[2] E.H. Hauge and E.G.D. Cohen, Phys. Letters 25A (1967) 78; J. Math. Phys. 10 (1969) 397.
[3] W.W. Wood and F. Lado, J. Comp. Phys. 7 (1971) 528.
[4] H. van Beyeren, to be published.