Cantor-Von Neumann Set-Theory

[Names suppressed]

MSC: 03E30, 00A30

key words: set, limitation, absolute infinity, ultimacy.

August 2009

length of this paper: about 5.000 words

Summary

In this elementary paper we establish a few results in set-theory. *Cantor-Von Neumann set-theory* (CVN) is a reformulation of Von Neumann's original theory of *functions and things* that does not introduce 'classes' (let alone 'proper classes'), developed in the 1920ies. In CVN, both the Axiom of Pair-Sets and 'half' the Axiom of Limitation are redundant — the last result is novel. Further we show, in contrast to how things are usually done, that some theorems, notably Pair-Sets, can be proved *without* invoking the Replacement Schema (F) and the Power-Set Axiom. Also the Axiom of Choice is redundant in CVN, because a stronger proposition of Global Choice is a theorem of CVN. The philosophical interest of CVN, which is very succinctly indicated, lies in the fact that it is far better suited than Zermelo-Fraenkel set-theory as an axiomatisation of what Hilbert once called *Cantor's Paradise*.

Contents

0 Introduction

In 1928, Von Neumann published his grand axiomatisation of Cantorian set-theory [1925; 1928]. Although Von Neumann's motivation was thoroughly Cantorian, he did not take the concept of a set and the membership-relation as primitive notions, but the concepts of a *thing* and a *function* — for reasons we do not go into here. This, and Von Neumann's cumbersome notation and terminology (II-things, II.I-things) are the main reasons why initially his theory remained comparatively obscure. Then came Paul Bernays [1937-1953; 1957]. He dressed up Von Neumann's theory in logicist *haute couture*, notably with *classes* (extensions of predicates), and cut out its Cantorian heart, the Axiom of Limitation (see below). And then, in 1938, came Gödel. He took this theory of *sets and classes* as the framework for proving his famous consistency results of the Axiom of Choice and the Generalised Continuum Hypothesis. Gödel also added the notion of a 'proper class' — as if extensions of predicates (classes) suddenly stop being extensions and become 'improper' when they happen to be sets too. The resulting theory of *sets and classes*, which is usually called 'Von Neumann-Bernays', 'Von Neumann-Bernays-Gödel' or even 'Gödel-Bernays' set-theory[1], thus became known and was used more and more, as time passed, by logicians and set-theoreticians; it has however remained little known among working mathematicians other than set-theoreticians or logicians. The standard axiomatisation still is *Zermelo-Fraenkel set-theory* (ZFC), glossing over possible qualms concerning the Axiom of Choice.[2]

Elsewhere we have argued that not ZFC, but what we propose to call *Cantor-Von Neumann set-theory* (CVN) is the best available axiomatisation of what Hilbert famously baptised "Cantor's Paradise".[3] The theory CVN results when Von Neumann's theory of functions and things is faithfully reformulated in the standard, 1st-order language of pure set-theory (denoted by $\mathcal{L}_\in$) extended with a single primitive set $\mathbf{V}$, and certain redundant axioms are deleted.

The purpose of the present note is to write down Cantor-Von Neumann set-theory formally (Section 2), to prove that 'one-and-a-half' axiom is redundant, and to prove some

[1]See Stoll [1963: 318], Fraenkel *et al.* [1973: 128], Jech [1978: 76], Mostowski in Müller [1976: 325], Enderton [1977: 10] and Kunen [1980: 35]. To add to the confusion, Fraenkel *et al.* [1973: 137] call what is almost our Cantor-Von Neumann set-theory '$G \wedge (*)$', where $(*)$ stands for the Axiom of Limitation (the language $\mathcal{L}_\in$ is then extended with 'class-variables' to language $\mathcal{L}_\in^*$). But this is *Von Neumann's* theory, not *Gödel's*!

[2]Cf. Fraenkel *et al.*'s overview [1973], Ch. II. Ironically, the name 'Zermelo-Fraenkel' is due to Von Neumann [1961: 321, 348], who also provided the (correct formulation of the) Axiom of Replacement, who added the Axiom of Regularity, and who created the canonical theory of ordinal and cardinal numbers; all of this is standardly transplanted to Zermelo's [1908] axiomatisation in order to obtain ZFC.

[3]Anonymous [2002]. This work gratefully builds on Hallett's seminal monograph [1984] on the philosophy and history of Cantorian set-theory.

axioms of ZFC in CVN in a manner that differs from the usual deductions (Section 3). The fact that half of the axiom of Limitation is redundant has gone unnoticed for about eighty years — as far as this author is aware of. But first, in order to have some idea what the *conceptual* watershed between ZFC and CVN consists in, and *a fortiori* to have a solid motivation for considering the theory CVN at all, we begin by providing a *very* succinct overview of this watershed (Section 1). We emphasise with all powers invested in us that *the subject of the present paper is not this conceptual watershed, but a few rigourous results that are the spin-off of an elaborate philosophical-foundational inquiry into Cantorian set-theory* (cf. Anonymous [2002]).

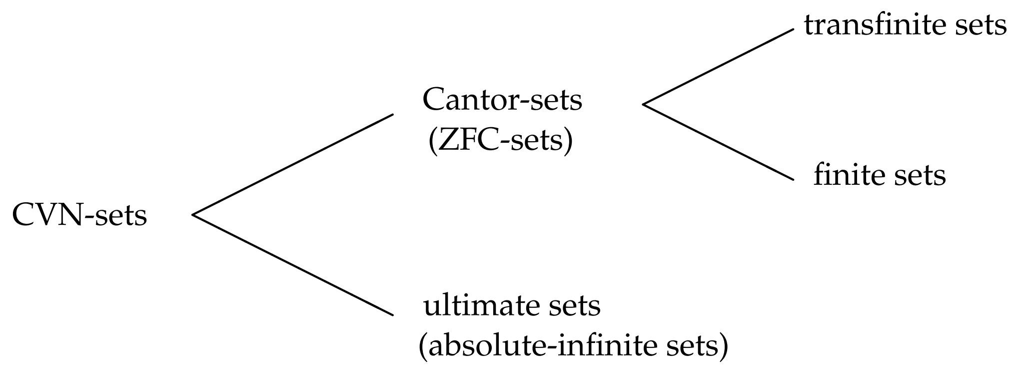
1 Zermelo-Fraenkel versus Cantor-Von Neumann

First comes a string of definitions (which we do not spell out formally[4]): a set is *potential-infinite* iff it sustains a linear ordering *and* for every value that a variable running over this set can assume, there is a larger value it can assume, *i.e.* the linear ordering has no top in the set; a set is *finite* iff it can be bijected to $\{1, 2, 3, \ldots, n\}$ for some $n \in \omega$, where ω is the first limit Von Neumann-ordinal; a set is *actual-infinite* iff a proper subset of it can be surjected onto it; a set is *ultimate* (Quine) iff no set has it as a member; a set is *absolute-infinite* iff it is equinumerous to the set of all non-ultimate sets (Von Neumann); a set is *increasable* iff it can be surjected onto some more inclusive set; a set is *transfinite* iff it is actual-infinite and increasable. A set is a *Cantor-set* (English translation of Cantor's German *Menge*) iff it is increasable, well-founded, well-orderable and has a unique ordinal number as soon as it is ordered well, and it has a unique cardinal number; and, finally, a set is *combinatorially inept* iff it does not arise in the cumulative hierarchy.

Standard *Zermelo-Fraenkel set-theory* (ZFC) is defined as the 1st-order deductive closure of the Axioms of Extensionality, Union, Power, Infinity, Replacement, Regularity and Choice; Replacement entails Separation, and Replacement and Power together entail Pairing.[5] *Cantor-Von Neumann set-theory* (CVN) is formulated in $\mathcal{L}_\in$ extended with a single 'logical constant' $\mathbf{V}$, a primitive *set* (call this language $\mathcal{L}_\in^{\mathbf{V}}$); we define CVN as the 1st-order deductive closure of the Axioms of Universe ($\mathbf{V}$ *includes* but does not *contain* every set), Extensionality, Power, Infinity, Set-Existence (every predicate with only bounded quantifiers, possibly with set-parameters, has a set-extension of non-ultimate members), Regularity and Weak Limitation (all ultimate sets are absolute-infinite). Von Neumann's Axiom of Limitation is the conjunction of Weak Limitation and its converse: exactly ulti-

[4]See Anonymous [2002] for the formal definitions and the proofs of the theorems we are about to assert — all proofs are easy.

[5]See Fraenkel *et al.* [1973: 22, 52], Lévy [1979: 23-24], Suppes [1960: 237], Stoll [1963: 304].



mate sets are absolute-infinite (cf. Section 2 for details).

One proves that in ZFC all sets are Cantor-sets and that neither absolute-infinite nor ultimate sets exist. In CVN one proves that a set is potential-infinite iff it is actual-infinite; that a set is ultimate iff it is absolute-infinite (Limitation) iff it is combinatorially inept; that a set is not ultimate iff it is a Cantor-set; that every set is either absolute-infinite or transfinite or finite; that every set is either ultimate or a Cantor-set; and that every set is either increasable or ultimate.

A.H. Kruse essentially proved that CVN is a conservative, hence an equiconsistent deductive extension of ZFC: every theorem of CVN in which only Cantor-sets occur already is a theorem in ZFC.[6] This means that the Cantor-sets of CVN coincide with the ZFC-sets (see picture). Thus in CVN one remarkably proves, rather than assumes (as in ZFC), that *every Cantor-set has a union-set and a choice-set*.[7] Further, CVN is finitely axiomatisable, in contradistinction to ZFC.

Now, various assertions of Cantor are *proved* in CVN, whereas they are *disproved* in ZFC, *e.g.* that the "actual-infinite has to be subdivided into the *increasable actual-infinite* and the *unincreasable actual-infinite*"[8] (which Cantor called the "transfinite" and the "absolute-infinite", respectively[9]); that the whole of all Cantor-sets is a "perfectly well-defined" absolute-infinite set[10]; that "every potential infinity presupposes an actual-infinity"[11]; that an absolute-infinite set is "mathematically indeterminable"[12] (when interpreted as

[6]Cf. Fraenkel *et al.* [1973: 136-137]. Kruse proved this for VN* (see Table); it then follows that it also holds for CVN.

[7]See Fraenkel *et al.* [1973: 137].

[8]Cantor [1932: 375].

[9]Cantor [1932: 405].

[10]Cantor [1932: 448].

[11]Cantor [1932: 410-411].

[12]Cantor [1932: 375].

combinatorial ineptitude) and "cannot be conceived of as a member of another set"[13] (they are ultimate); that absolute-infinite sets "have to be admitted and acknowledged"[14] (they exist); and more. All in all, CVN provides a rigourous *legalisation* for a host of informal claims of Cantor, whereas ZFC *outlaws* them. For this reason, CVN is far better suited to be the axiomatisation of Cantor's paradise than ZFC.

So much for a brief comparison between ZFC and CVN. We refer to Anonymous [2002] for an elaborate inquiry into CVN, its heuristics and motivation, and into how it compares with ZFC conceptually.

2 Axiomatics

To rehearse, the 1st-order formal language $\mathcal{L}_\in$ of pure set-theory has only *set-variables* ($A, B, C, D, F, \ldots X, Y, Z$; occasionally we use m, n, p as finite-ordinal-variables), and the membership-relation ($\in$) as its only predicate-constant; then $\ulcorner X \in Y \urcorner$ is its only type of atomic sentence. In the language of CVN, denoted by $\mathcal{L}_\in^{\mathbf{V}}$, we have in addition to $\mathcal{L}_\in$ one primitive *set* $\mathbf{V}$. Identity between two sets ($X = Y$) is defined in Hilbert-Bernays fashion as the conjunction of sharing all members ($A \in X \longleftrightarrow A \in Y$) and being shared as members by all the same sets ($X \in B \longleftrightarrow Y \in B$). We use $\equiv$ for term-definition and $\Longleftrightarrow$ for sentence- and predicate-definition.

Throughout we assume that all the usual definitions are in force (power-set $\wp X$ of set X, union-set $\bigcup X$ of set X, the empty set $\varnothing$, *etc.*; see Fraenkel *et al.* [1973], Chapter II). We emphasise that $\mathcal{L}_\in^{\mathbf{V}}$ does *not* contain Bernaysian 'classes', Quinean 'virtual sets' or Gödelian 'proper classes'. We use bold-faced capitals for ultimate set-names (Kunen-convention); these sets are just as 'real' as all their non-ultimate siblings. We next spell out the axioms of CVN formally.

The Universe Axiom says that $\mathbf{V}$ includes every single set:

$$\text{(Univ)} \quad \forall X : \; X \subseteq \mathbf{V} \, . \tag{1}$$

So $\mathbf{V}$ is the 'domain of discourse' of CVN *in the sense that* $\mathbf{V}$ includes every set whose existence can be demonstrated. We introduce ultimacy formally:

$$\text{Ultim}(X) \iff \forall Y : X \notin Y \, . \tag{2}$$

According to Universe (1), the members of any set, ultimate or not, are also members of $\mathbf{V}$; ultimate sets are by definition (2) not members of $\mathbf{V}$; hence a set is not ultimate iff it is

[13] Grattan-Guinness [1971: 119].
[14] Cantor [1932: 205].

a member of $\mathbf{V}$, or which is the same, a set is ultimate iff it is (a subset but) not a member of $\mathbf{V}$:

$$\text{Ultim}(X) \longleftrightarrow X \notin \mathbf{V} . \tag{3}$$

In other words, $\mathbf{V}$ is the set of exactly the non-ultimate sets.

The Axiom of Set-Existence (SetEx) asserts that for any predicate $\boldsymbol{\varphi}(\cdot, Y)$, which may have any number of arbitrary set-parameters $Y_1, Y_2, \ldots, Y_n$ (abbreviated by Y), and which does not contain unbounded quantifiers but may contain quantifiers running over *all* non-ultimate sets (collected in $\mathbf{V}$), there exists a set S of all non-ultimate sets for which the predicate holds:

$$\text{(SetEx)} \quad \forall\, Y,\ \exists\, S \subseteq \mathbf{V},\ \forall\, X \in \mathbf{V} \left(X \in S \longleftrightarrow \boldsymbol{\varphi}(X, Y) \right) . \tag{4}$$

Notice that in general the set-extension S of SetEx (4) can be *ultimate*, but all its members, the sets that fall under the predicate, are not ultimate (they cannot be because they are *members*); whether set S is ultimate or not is something we have to prove on the basis of the other axioms. Further, the fact that the variable X in (4) is bounded to $\mathbf{V}$ makes it possible to reduce this list of denumerable many axioms to a eight axioms.[15] The restriction to bounded quantifiers in Set-Existence betrays that it has a whiff of predicativity in it — but certainly not more than a whiff, because quantification over *all* sets in $\mathbf{V}$ still is light-years removed from Russellian predicativity or Quinean stratification.

For convenience we define $\mathbf{V}_\varnothing$ as the set of all non-ultimate, non-empty sets (Set-Existence):

$$\mathbf{V}_\varnothing \equiv \{ X \in \mathbf{V} \mid \exists\, Y \in \mathbf{V} : Y \in X \} . \tag{5}$$

That *some* set exists is a theorem of logic; this set may be $\mathbf{V}$ and $\mathbf{V}$ may be empty; to prove that $\mathbf{V} \neq \varnothing$, another axiom besides SetEx is needed, such as Infinity (which asserts the existence of ω); the existence of $\varnothing$ as a non-ultimate set follows from Infinity, because $\varnothing \in \omega$. Next come the familiar axioms of Extensionality, Pair, Union, Power, Infinity, Separation and Regularity (there is implicit universal quantification over all variables

[15] Fraenkel *et al.* [1973: 129-130].

occurring free).

$$
\begin{array}{ll}
\text{(Ext)} & (X \subseteq Y \wedge Y \subseteq X) \;\longrightarrow\; X = Y\,. \\
\text{(Pair)} & X, Y \in \mathbf{V} \;\longrightarrow\; \{X, Y\} \in \mathbf{V}\,. \\
\text{(Union)} & X \in \mathbf{V} \;\longrightarrow\; \bigcup X \in \mathbf{V}\,. \\
\text{(Pow)} & X \in \mathbf{V} \;\longrightarrow\; \wp X \in \mathbf{V}\,. \\
\text{(Inf)} & \omega \in \mathbf{V}\,. \\
\text{(Sep)} & Z \in \mathbf{V},\; Y \in \mathbf{V},\; \exists\, A \in \mathbf{V},\; \forall\, X \big(X \in A \;\longleftrightarrow\; X \in Z \wedge \boldsymbol{\varphi}(X, Y) \big)\,. \\
\text{(Reg)} & \forall\, X \in \mathbf{V},\; \exists\, Y \in \mathbf{V}\, (Y \in X \wedge Y \cap X = \varnothing)\,.
\end{array}
\tag{6}
$$

SetEx (4) provides us already with pair-sets, union-sets and power-sets, and even with ω, but nothing can be said as to whether these sets are ultimate or not; the Axioms of Pair, Union, Power and Infinity decide this by asserting that these sets are *not* ultimate.

We employ the usual definitions of a *function* F from its *domain* D to co-domain C, and of *the range of* F, denoted by $F : D \to C$ and R_F, respectively ($R_F \subseteq C$). The Axiom of Replacement then reads that for every function F from domain D to co-domain C it holds that if its domain is not ultimate, then neither is its range ($R_F \equiv F[D] \subseteq D$):

$$
\text{(F)} \quad (F : D \to C) \;\longrightarrow\; \big(D \in \mathbf{V} \;\longrightarrow\; R_F \in \mathbf{V} \big)\,. \tag{7}
$$

The Axiom of Global Choice reads that there is some function $F \subset \mathbf{V}$ (also called a 'choice-function') that sends every non-empty set to a member of it:

$$
\text{(GChoice)} \quad \exists\, F \subset \mathbf{V} \big(F : \mathbf{V}_\varnothing \to \mathbf{V}_\varnothing \;\wedge\; \forall\, X \in \mathbf{V}_\varnothing : F(X) \in X \big)\,. \tag{8}
$$

GChoice (8) implies Choice as we know it from ZFC: restrict F in (8) to non-ultimate subsets of $\mathbf{V}_\varnothing$.

We now arrive at the Cantorian heart of CVN. Von Neumann essentially proposed two precise renditions of Cantor's idea of an 'absolute-infinite' set: not as an actual-infinite set that cannot be *increased* (*i.e.* there is a more inclusive set onto which it cannot be surjected), but, first, as a set that *cannot be collected further into any other set*. This is ultimacy (2). Von Neumann's second definition of 'absolute-infinite' reads as follows: a set is *absolute-infinite* iff it is equinumerous to the set $\mathbf{V}$ of all non-ultimate sets:

$$
\text{AbsInf}(X) \iff X \sim \mathbf{V}\,, \tag{9}
$$

where $\sim$ is the 'equinumerosity-relation'. Definition: set X is *equinumerous* to set Y iff there is 'bijection' between them:

$$
X \sim Y \iff \exists\, F \subset \mathbf{V} : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y\,, \tag{10}
$$

where a *bijection* F *from* X *to* Y, denoted by $F : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$, is defined as a function whose range Y is such that every member of Y comes from exactly one domain-member:

$$F : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y \iff \big(F : X \to Y \ \wedge\ \forall B \in Y\ \exists! A \in X : \langle A, B \rangle \in F \big) . \tag{11}$$

Thus for Von Neumann, 'too big' means 'biggest': the only way for a set to become absolute-infinitely big is to be as big as possible. Since Von Neumann intended ultimacy as a new way of looking at absolute-infinity, his Axiom of Limitation expresses precisely this: the ultimate sets and the absolute-infinite are co-extensive:

$$\text{(Lim)} \quad \text{Ultim}(X) \longleftrightarrow \text{AbsInf}(X) . \tag{12}$$

The Weak Axiom of Limitation asserts one conjunct of Limitation (12), namely that *all ultimate sets are absolute-infinite*:

$$\text{(WkLim)} \quad \text{Ultim}(X) \longrightarrow \text{AbsInf}(X) . \tag{13}$$

For the sake of reference and overview, we define the following theories, as the 1st-order deductive closures of the axioms in the language mentioned:

Theory	Language	Axioms	Theorems
ZFC	$\mathcal{L}_\in$	Ext, Inf, Reg, Pow, Un, F, Choice	Sep, Pair
CVN_0	$\mathcal{L}_\in^{\mathbf{V}}$	SetEx, Ext, Inf, Reg, Univ	
CVN	$\mathcal{L}_\in^{\mathbf{V}}$	SetEx, Ext, Inf, Reg, Univ, Pow, WkLim	GChoice, F, Sep, Un, Pair, Lim
VN^*	$\mathcal{L}_\in^*$	SetEx, Ext, Inf, Reg, Pow, Un, Pair, Lim	GChoice, F, Sep (Un, Pair)

VN^* comes closest to Von Neumann's original theory (VN), when reformulated in $\mathcal{L}_\in$ enriched with 'class-variables' in Bernaysian fashion ($\mathcal{L}_\in^*$)[16]; Union and Pair between brackets in the 'Theorems'-column indicate they need not be taken as axioms, as Von Neumann originally did, because they can be proved on the basis of the other axioms. For reasons indicated in Section 1, we baptise the theory in the third line of the Table above *Cantor-Von Neumann set-theory* (CVN).

The Axiom of Limitation (12) provokes the question whether it is not some philosophical ornament, solely put forward by Von Neumann to propitiate the Cantorian spirit. The answer is a resounding denial, for Limitation is, in the presence of CVN_0 plus Pow and

[16] Fraenkel *et al.* [1973] call VN^* *without* Limitation 'Von Neumann-Bernays' (VNB) and with Choice 'VNBC'.

Pair, equivalent to the conjunction of Global Choice, Separation, Replacement (this was proved by Von Neumann [1928]) and Union (proved by Lévy [1968])! This is one excellent reason why Von Neumann adopted the theory VN (with Union), because then Global Choice, Separation and Replacement become *theorems*; consequently one then finds ZFC among its deductive offspring so that VN deductively extends ZFC. Thus calling the Axiom of Limitation, perhaps pejoratively, a 'Cantorian ornament' does not even begin to do justice to it. Besides its deductive strength, Von Neumann motivated the axiom on two independent grounds: (i) it captures Cantor's notion of an "unincreasable, actual-infinite set" and "recognises and admits their existence" (all Cantor's words); and (ii) it blocks the deduction of the well-known antinomies (Russell, Burali-Forti) and simultaneously, seemingly *per impossibile*, it almost saves the Peano-Frege principle of full comprehension according to which *every* predicate has an extension (by binding variables mildly to **V**).

From the axioms we turn to the theorems.

3 Weak Limitation and its Consequences

3.0 By means of a series of theorems in 3.1-3.5 we shall work our way to the central result: CVN entails VN, which is to say that Pair and the converse of Weak Limitation can be proved in CVN:

$$\text{CVN} \vdash \text{VN}^* . \tag{14}$$

Of course, as soon as we have Limitation, we have Union, Separation, Replacement and (Global) Choice, due to Von Neumann's result mentioned earlier, but it is interesting to know, we believe, that several of these theorems can be proved *without* invoking Limitation.

3.1 We first prove that **V** is an ultimate set by using Universe, Pow, Set-Existence and Regularity:

$$\text{CVN}_0,\ \text{Pow} \vdash \text{Ultim}(\mathbf{V}) . \tag{15}$$

Proof. (We point out that Infinity will not be employed.) We prove it by *reductio ad absurdum*. Assume that **V** is not ultimate (Reductio Assumption; henceforth: RA).

To steer at a contradiction, we remark that RA entails that **V** is self-membered because **V** contains by theorem (3) *all* non-ultimate sets — an implication of Univ (1):

(i) $\mathbf{V} \in \mathbf{V}$.

Pow guarantees that the singleton-set $\{X\}$ of every non-ultimate set $X \in \mathbf{V}$ (whose existence follows from SetEx) is not ultimate either, because $\{X\} \in \wp\wp X \subseteq \mathbf{V}$. Then by (RA) $\{\mathbf{V}\} \in \mathbf{V}$, because of theorem (3). We obviously also have

(ii) $\mathbf{V} \in \{\mathbf{V}\}$.

Regularity, when applied to non-ultimate set $\{\mathbf{V}\}$, now says there is some $Y \in \{\mathbf{V}\}$ such that $\{\mathbf{V}\}$ and Y have no members in common. This Y must be $\mathbf{V}$, because $\mathbf{V}$ is the only member of $\{\mathbf{V}\}$. So $\mathbf{V}$ and $\{\mathbf{V}\}$ have no members in common. But according to (i) and (ii) they do have a member in common: $\mathbf{V}$. Contradiction. □

3.2 Pair is needed to make ordered pairs and hence to make Cartesian product-sets, relations, functions, operations; it is a *conditio sine qua non* for set-theory. In ZFC, Pair (6) is proved on the basis of Pow and Replacement. Here we prove it without using any of these axioms. We prove it on the basis of CVN_0 enriched with WkLim (see Table):

$$\mathrm{CVN}_0,\ \mathrm{WkLim} \vdash\ A, B \in \mathbf{V} \ \longrightarrow\ \{A, B\} \in \mathbf{V}\ . \tag{16}$$

Proof. Let $A, B \in \mathbf{V}$. By Set-Existence and Extensionality, the pair-set $\{A, B\} \subseteq \mathbf{V}$ (Univ) exists uniquely. To prove that it is a member of $\mathbf{V}$, assume, for *reductio*, that $\{A, B\}$ is not a member of $\mathbf{V}$, *i.e.* that it is ultimate (RA).

Then $\{A, B\}$ is equinumerous to $\mathbf{V}$, due to Weak Limitation (13). Let f be the bijection from $\mathbf{V}$ onto $\{A, B\}$, the existence of which is by definition (10) logically equivalent to the equinumerosity of $\{A, B\}$ and $\mathbf{V}$. Well, Infinity gives us $\omega \in \mathbf{V}$. Then $\omega \subseteq \mathbf{V}$. Let $m, n \in \omega$ be distinct. If $f(m)$ or $f(n)$ is not equal to A or B, then f is not even a function with range $\{A, B\}$ and we already have a contradiction. To avoid this contradiction, we must have that $f(m) = A$ and $f(n) = B$ — or with A and B interchanged. Obviously, there is always a $p \in \omega$ which is distinct from both n and m. There *must* be a member of $\{A, B\}$, say Z, distinct from A and B, such that $f(p) = Z$. But $\{A, B\}$ has no other members besides A and B. Contradiction. □

Now that we have Pair in CVN_0 plus WkLim, we can make pairs, ordered pairs and sets of ordered pairs, *etc*. All functions, relations and operations are now available. Notice that neither Replacement nor Pow was needed to prove Pair, whereas in ZFC precisely these, and only these, axioms are used to prove Pair — these proofs do not carry over to CVN because we do not have Replacement (yet) and our proof of Replacement (as well as Von Neumann's) requires the presence of Pair.[17] Instead of Inf, Power can be used in

[17] Von Neumann had Pair as an axiom [1928; 1961: 344]. To prove Pair from Replacement and Pow, con-

the proof above to produce more than 2 members of **V** via $\wp A$, $\wp B$, $\wp\wp A$, $\wp\wp B$ etc. so as to get at a contradiction.

3.3 The absolute-infinity of **V** is an immediate consequence of Theorem (15) as soon as we have Weak Limitation. But it can be proved *without* explicitly appealing to Weak Limitation, on the basis of Universe, Extensionality, Set-Existence and Pair.

$$\text{Univ, Ext, SetEx, Pair} \vdash \text{AbsInf}(\mathbf{V}) \,. \tag{17}$$

This means we have the absolute infinity of **V** in CVN_0 plus WkLim as a consequence of the Pair Theorem (16).

Proof. We prove something stronger: *every set is equinumerous to itself*. First we define for an arbitrary set $X \subseteq \mathbf{V}$, the identity $I_X \subseteq \mathbf{V}$ as the 'diagonal set' of ordered pairs $\langle A, A\rangle$, for all $A \in X$. The unique existence of I_X is guaranteed by Set-Existence, Extensionality and Pair. According to definition (10), X is equinumerous to itself iff there is a bijection between set X and itself, *i.e.* there is a set of ordered pairs $\langle A, B\rangle$, where $A, B \in X$, such that every $A \in X$ and every $B \in X$ occur exactly once. Clearly the diagonal set I_X qualifies as such a set.

The equinumerosity of **V** with itself yields its absolute-infinite character by definition (9). $\square$

As a corollary of Theorems (15) and (17) we have an instance of the Axiom of Limitation for **V** (theorem of logic: $\boldsymbol{\psi} \wedge \boldsymbol{\varphi}$ entails $\boldsymbol{\psi} \longleftrightarrow \boldsymbol{\varphi}$):

$$\text{CVN}_0,\ \text{WkLim} \vdash \text{Ultim}(\mathbf{V}) \longleftrightarrow \text{AbsInf}(\mathbf{V}) \,. \tag{18}$$

3.4 Now we prove Zermelo's Axiom Schema of Separation, which says that every predicate has a set-extension of members in a given set.

$$\text{CVN}_0,\ \text{Pow} \vdash Z \in \mathbf{V},\ Y \in \mathbf{V},\ \exists\, A \in \mathbf{V},\ \forall\, X \left(X \in A \ \longleftrightarrow\ X \in Z \wedge \boldsymbol{\varphi}(X, Y)\right). \tag{19}$$

Proof. (It serves to mention that of CVN_0 neither Regularity nor Infinity will be used.) Let Z be an arbitrary non-ultimate set, $Z \in \mathbf{V}$, and $\boldsymbol{\varphi}(\cdot, Y)$ some sentence with one free variable and n set-parameters, abbreviated by Y. Set-Existence gives us the set-extension A of predicate $\ulcorner\boldsymbol{\varphi}(X, Y) \wedge X \in Z\urcorner$, which has Z as an additional non-ultimate set-parameter. Is A a non-ultimate set? Yes, if we can produce a set that contains A. Since

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sider $\varnothing$, which exists according to Separation (a consequence of Replacement); make $\wp\wp\varnothing = \{\{\varnothing\}, \varnothing\}$, which is a non-ultimate set (Pow); biject it to the set $\{A, B\}$; and finally invoke Replacement to conclude that $\{A, B\}$ is not ultimate either.

$A \subseteq Z$, the power-set $\wp Z \in \mathbf{V}$ is such a set: $A \in \wp Z$. $\square$

The Axiom Schema of Separation can be replaced with a single axiom (notably this *cannot* be done in ZFC, where $\mathbf{V}$ is not available): a subset of a non-ultimate set is not ultimate; formally,

$$X \in \mathbf{V} \;\longrightarrow\; (X \cap \mathbf{V}) \in \mathbf{V}\,. \tag{20}$$

This can also be directly proved, by first noticing that, given $\mathbf{V}$ (3), it is a theorem of logic that $(X \cap \mathbf{V}) \subseteq X$, and then noticing that $X \cap \mathbf{V}$ is not ultimate because $(X \cap \mathbf{V}) \in \wp X$.

3.5 Definition: X is *minumerous or equinumerous to* Y, denoted by $X \preccurlyeq Y$, iff X can be bijected to a subset of Y; and X is *minumerous to* Y, or synonymously, Y is *amplinumerous to* X, denoted by $X \prec Y$, iff X is minumerous or equinumerous to Y and Y is not minumerous or equinumerous to X:

$$\begin{aligned} X \preccurlyeq Y \quad &\Longleftrightarrow \quad \exists\, Y' \subseteq Y,\ \exists\, F \subseteq \mathbf{V} : X \rightarrowtail\!\!\!\twoheadrightarrow Y'\,. \\ X \prec Y \quad &\Longleftrightarrow \quad X \preccurlyeq Y \ \wedge\ \neg(Y \preccurlyeq X)\,. \end{aligned} \tag{21}$$

We report two theorems: Cantor's Power Theorem, according to which every set is minumerous to its power-set, and the Cantor-Dedekind-Bernstein 'Minumerosity Theorem', which asserts the a-symmetry of the relation $\preccurlyeq$:

$$\begin{aligned} &\mathrm{CVN}_0,\ \mathrm{Pow}\ \vdash\ X \prec \wp X \\ &\mathrm{CVN}_0,\ \mathrm{Pow}\ \vdash\ (X \preccurlyeq X\ \wedge\ Y \preccurlyeq X) \;\longleftrightarrow\; X \sim Y\,. \end{aligned} \tag{22}$$

The following theorem directly follows from the definition of minumerosity (21) and theorem (22):

$$\begin{aligned} \mathrm{CVN}\ \vdash\quad X \prec Y \quad &\longleftrightarrow\ \big(\neg(Y \preccurlyeq X)\ \wedge\ X \not\sim Y\big) \\ &\longleftrightarrow\ \big(X \preccurlyeq Y\ \wedge\ X \not\sim Y\big)\,. \end{aligned} \tag{23}$$

Cantor's Power Theorem (22) can be proved on the basis of SetEx (Separation suffices), Ext, Pow and Pair, which together yield that (a) $X \preccurlyeq \wp X$ (easy: $A \mapsto \{A\}$ bijects X onto a subset of $\wp X$); and (b) $X \not\sim \wp X$ (by means of a *reductio* argument); in the final step of the proof, the Minumerosity Theorem is invoked, via version (23), to deduce from (a) and (b) that $X \prec \wp X$.[18] The Minumerosity Theorem can be proved from Sep, Ext and Pair, hence also in CVN, which has Sep (19) and Pair (16) as theorems.

[18] See the proofs of Cantor's Power Theorem and the Minumerosity Theorem in, for instance, Stoll [1963: 81-82, 86], Lévy [1979: 85, 87].

Now we are in a position to prove the converse of Weak Limitation in CVN: *every absolute-infinite set is ultimate*:

$$\text{CVN} \ \vdash \ \text{AbsInf}(X) \ \longrightarrow \ \text{Ultim}(X) \ . \tag{24}$$

Proof. Let X be an absolute-infinite set: $X \sim \mathbf{V}$ (Premise). We have to prove that X is ultimate. The *reductio* assumption is that X is *not* ultimate: $X \in \mathbf{V}$ (RA).

Then $\wp X \in \mathbf{V}$ (Pow). Every set $Y \in \mathbf{V}$ can be bijected to a subset of $\mathbf{V}$, namely to itself by means of the identity; hence $Y \preccurlyeq \mathbf{V}$. In combination with $Y \prec \wp Y$ (22) we then deduce that $X \prec \mathbf{V}$. By means of (23) we conclude that $\mathbf{V} \not\sim X$, which contradicts the Premise. □

Every axiom of CVN is invoked to prove that absolute-infinite sets are ultimate (24), all via the Pair Theorem (16) and the Minumerosity Theorem (22). We then arrive at:

$$\text{CVN} \ \vdash \ \text{Lim} \ , \tag{25}$$

from which, in combination with the Pair Theorem, our main result follows: CVN entails VN (14).

3.6 The Axiom of Replacement, which in ZFC is an axiom schema, is reduced in CVN to a single sentence of $\mathcal{L}_\in$: if the domain of a function is not-ultimate, then neither is its range (7). We prove it on the basis of CVN; we submit the proof as a simpler one than Von Neumann's proof [928; 1961: 365]; only WeakLim will be involved.

$$\text{CVN} \ \vdash \ D_F \in \mathbf{V} \ \longrightarrow \ R_F \in \mathbf{V} \ . \tag{26}$$

Proof. Let F be a function whose domain D_F is not ultimate: $D_F \in \mathbf{V}$ (Premise).

Define the set of members of D_F which F sends to a given member $Y \in R_F$ (SetEx, Ext):

$$[Y]_F \ \equiv \ \{X \in D_F \mid F(X) = Y\} \ . \tag{27}$$

Since $[Y]_F \subseteq D_F$, and thus $[Y]_F \in \wp D_F$ (Pow), we may conclude that $[Y]_F$ is not ultimate (for every $Y \in R_F$). We next collect them in a set (SetEx, Ext):

$$Z_F \ \equiv \ \big\{[Y]_F \in \wp D_F \mid Y \in R_F\big\} \ .$$

Then $Z_F \subseteq \wp D_F$, which implies that

(i) $\quad Z_F \preccurlyeq \wp D_F$.

The range R_F is equinumerous to Z_F, because $Y \mapsto [Y]_F$ is bijective from R_F to Z_F. When we combine $R_F \sim Z_F$ with (i), we obtain that $R_F \preccurlyeq \wp D_F$. From this and $\wp D_F \prec$

$\wp\wp D_F$ (22) and $\wp\wp D_F \preccurlyeq \mathbf{V}$ (because $\wp\wp D_F \subseteq \mathbf{V}$), we then have that $R_F \prec \mathbf{V}$. By virtue of (23), we then have

(ii) $\neg(\mathbf{V} \preccurlyeq R_F)$.

If R_F were equinumerous to $\mathbf{V}$, then we would trivially have that $\mathbf{V} \preccurlyeq R_F$, in contradiction to (ii); hence $\mathbf{V}$ is not equinumerous to R_F. But then, by WkLim (13), R_F is not ultimate either. □

3.7 Both Lévy's proof of Union (6) as well as Von Neumann's proof of Global Choice (8) crucially employ Replacement, which only needs Weak Limitation.[19] To go *from* the *ultimacy* of the set $\mathbf{\Omega}$ of all ordinals *to* the equinumerosity of $\mathbf{\Omega}$ and $\mathbf{V}$, as Lévy does in his proof, one evidently needs Weak Limitation; and to go *from* the minumerosity of the union-set $\bigcup X$ in comparison with $\mathbf{V}$, hence from its implied non-equinumerosity in comparison with $\mathbf{V}$, *to* the non-ultimacy of $\bigcup X$, again Weak Limitation suffices. Also in the proof of Global Choice, where the equinumerosity of $\mathbf{\Omega}$ and $\mathbf{V}$ is concluded from the ultimacy of $\mathbf{\Omega}$, so that all sets can be well-ordered — so that by implication every set has a choice-set — only Weak Limitation is involved. We thus have without Limitation:

$$\text{CVN} \vdash \text{Union} \wedge \text{GChoice} . \tag{28}$$

3.8 The main conclusions of this paper are, besides that Pair is redundant axiom in Cantor-Von Neumann set-theory (CVN), that, first, 'half' of Von Neumann's Axiom of Limitation, which is the Cantorian heart of CVN, is redundant; and secondly, that the proofs of the important theorems in CVN (Pair, Separation, Replacement, Global Choice, Union) reveal that this 'half' of Limitation (we called Weak Limitation) is the part of Limitation that performs all the deductive labour. The strength of Weak Limitation is Herculean.

[19] Lévy [1968], Neumann [1928: 398-399].

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